Determination of Chapman and Vary-Chap variable scale-height profiles by inversion of vertical electron density profiles

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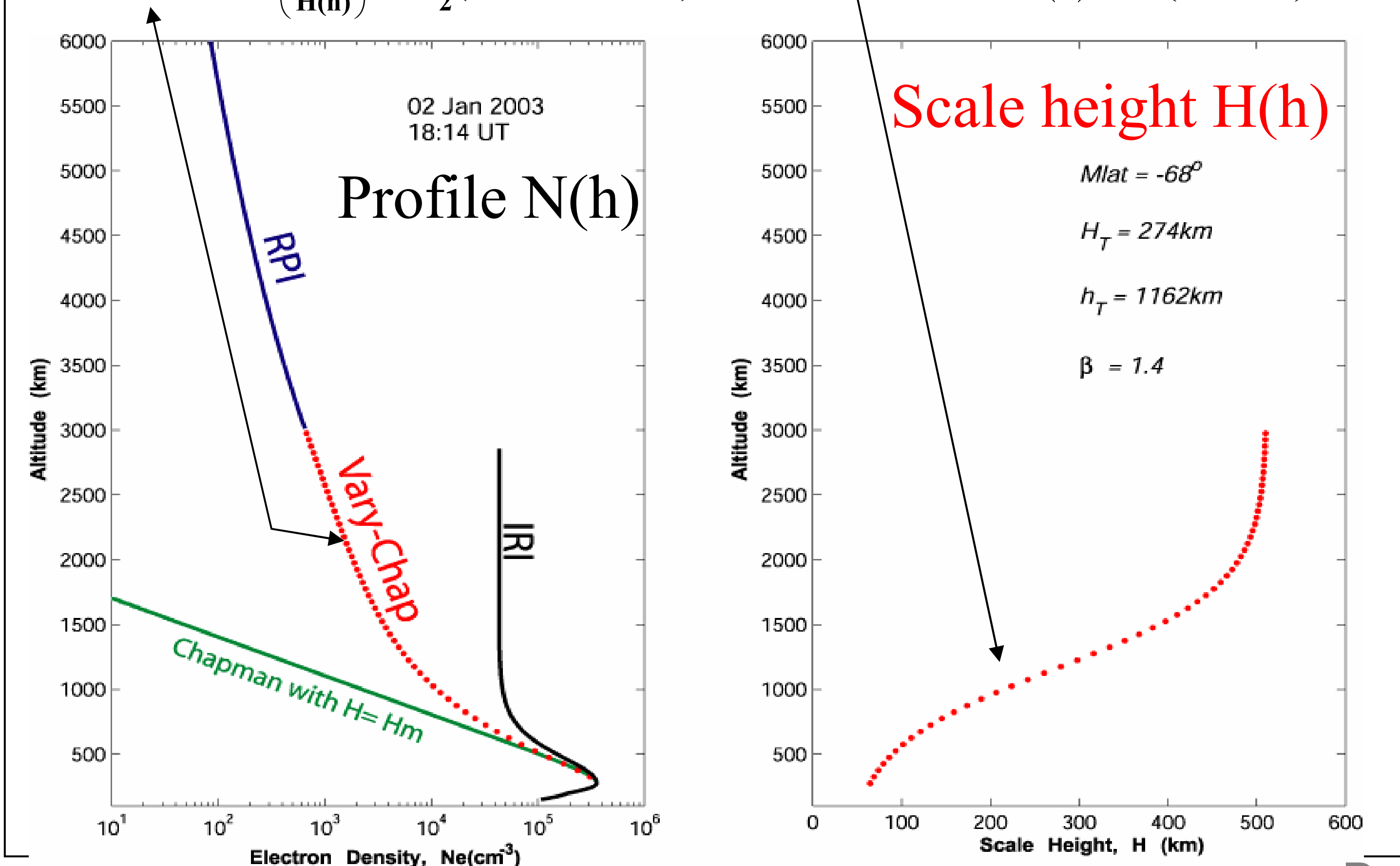
Overview

- Motivation
- Generalizations of Chapman Layer Functions
- Direct Determination of Variable Scale Heights
- Conclusions

Example: IRI Extension to Topside

$$N_{VC}(h) = N_m \left(\frac{H_m}{H(h)}\right)^{1/2} \exp \frac{1}{2}\left\{1 - y(h) - \exp\left[-y(h)\right]\right\}$$

$$H(h) = H_T + \frac{H_m - H_T}{\tanh(\beta)} \tanh\left(\beta \frac{h - h_T}{h_m - h_T}\right)$$



D. Bilitza

(α-)Chapman Layer Function

$$n(h) = N_0 \exp\left\{\frac{1}{2}\left(1 - y - \sec\chi \cdot e^{-y}\right)\right\} \qquad y = \frac{h - h_0}{H}$$

- Describes layers in the ionosphere caused by solar radiation
- Parameters:
  - N_0 … maximum ionization
  - H_0 … scale height
  - h_0 … height of maximum ionization
  - χ … solar zenith angle
- It is a good functional form for modelling ionospheric layers even at night when there is no solar radiation
- Chapman layer functions are used in
  - International Reference Ionosphere model
  - Ionosonde electron density reconstructions

Generalized Chapman Layer Functions/1

- In the following we set *sec χ = 1*
- We now allow the scale height H to depend on height:

$$H \rightarrow H(h) \qquad y = \int_{h_0}^{h} \frac{\mathrm{d}h'}{H(h')}$$

- Rationale: scale height changes with temperature and composition of the ionospheric plasma

$$H = kT/mg$$

- For $H(h)=H_0$ we recover the usual Chapman layer function
- These generalized Chapman layer functions are used in
  - IRI model extension to topside
  - Ionosonde electron density reconstruction

Generalized Chapman Layer Functions/2

1. Generalized Chapman layer function

$$n_{GC}(h) = N_0 \exp\left\{\frac{1}{2}\left(1 - y - e^{-y}\right)\right\} \qquad y = \int_{h_0}^{h} \frac{\mathrm{d}h'}{H(h')}$$

2. Vary-Chap function

$$n_{VC}(h) = N_0 \left(\frac{H_0}{H(h)}\right)^{1/2} \exp\left\{\frac{1}{2}\left(1 - y - e^{-y}\right)\right\} \qquad y = \int_{h_0}^{h} \frac{\mathrm{d}h'}{H(h')}$$

Maximum of electron density at height h_0: $N_0 = n(h_0) \quad H_0 = H(h_0)$

Reinisch, B. W.; Nsumei, P.; Huang, X.; Bilitza, D. K.
Modeling the F2 topside and plasmasphere for IRI using IMAGE/RPI and ISIS data

Given n(h) is it possible to *analytically* obtain H(h)?

Inversion

Both functions can be inverted:

- By solving for *y(h)* using Lambert W function technology

$$\tilde{n}_{GC}^{2}(h) = \exp\left(-y - e^{-y}\right) \qquad \tilde{n}_{GC}(h) = \frac{n_{GC}(h)}{N_0\sqrt{e}} \quad [1]$$

- By solving the differential differential equation

$$\tilde{n}_{VC}^{2}(h) = \frac{\mathrm{d}y}{\mathrm{d}h} \cdot \exp\left(-y - e^{-y}\right) \qquad \tilde{n}_{VC}(h) = \frac{n_{VC}(h)}{N_0\sqrt{eH_0}} \quad [m]$$

The Lambert W Function

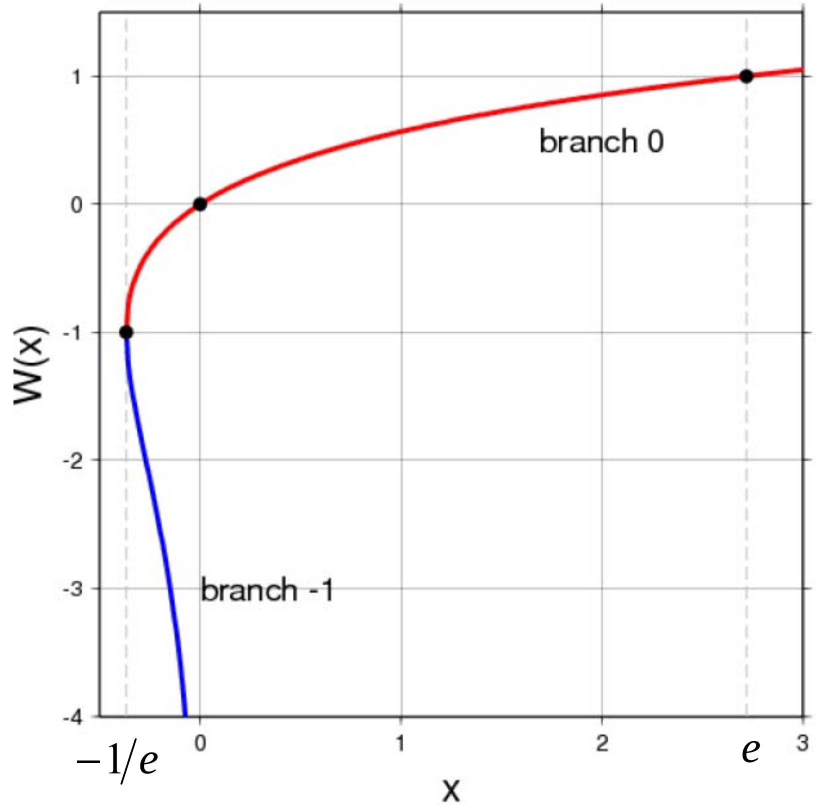
Implicit definition of the Lambert W function: $W(z)\cdot e^{W(z)} = z \quad , \quad z \in \mathbb{C}$

- is multi-valued
- has two real branches
- special values:

z	*W(z)*
0	*0*
-1/e	*-1*
e	*1*

- derivative:

$$\frac{\mathrm{d}W(z)}{\mathrm{d}z} = \frac{1}{e^{W(z)} + z}$$



Johann Heinrich Lambert
* 26. August 1728 in Mülhausen
† 25. September 1777 in Berlin

Corless et al. "On the Lambert W function"
Adv. Computational Maths. 5, 329 - 359 (1996)

Solution for n_{GC} / 1

1. Determine the maximum ionization N_0
2. Then compute

$$y(h) = -\ln\left(-\tilde{n}_{GC}^2(h)\right) - W\left(-\tilde{n}_{GC}^2(h)\right)$$

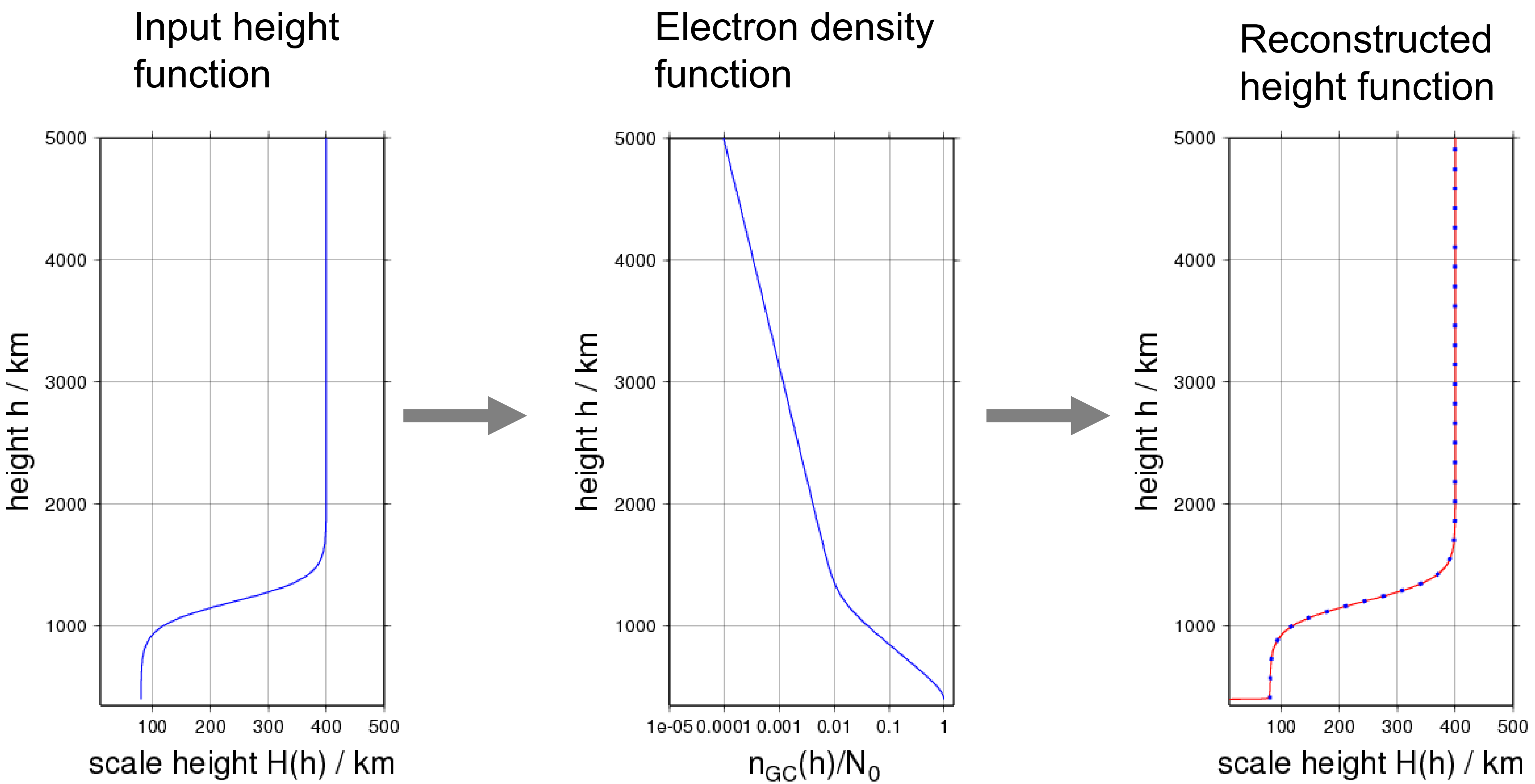
- Complication: depending on which branch of the W function we are, H(h) can become positive or negative (W is multivalued!):

$$H(h) = \left(\frac{\mathrm{d}y(h)}{\mathrm{d}h}\right)^{-1} = -\frac{\tilde{n}_{GC}}{2\cdot\tilde{n}_{GC}'}\cdot\left(1+W\left(-\tilde{n}_{GC}^2\right)\right)$$

- Branch selection:

$$\begin{cases} \dfrac{\mathrm{d}\tilde{n}_{GC}(h)}{\mathrm{d}h} > 0: & \text{use branch -1} \\ \dfrac{\mathrm{d}\tilde{n}_{GC}(h)}{\mathrm{d}h} < 0: & \text{use branch 0} \end{cases}$$

Solution for n_{GC} / 2



Solution for n_{GC} / 3

- The solution is numerical unstable around the maximum electron density:

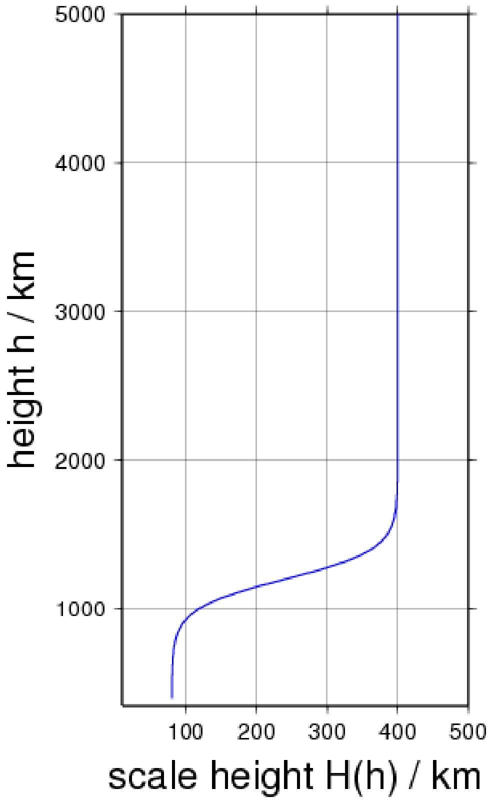
$$H(h_0) = -2 \cdot \frac{\tilde{n}_{GC}(h_0)}{\tilde{n}_{GC}'(h_0)} \cdot \left(1 + W\left(-\tilde{n}_{GC}^2(h_0)\right)\right) \overset{l'Hopital^2}{=} -\frac{N_0}{2n''}\bigg|_{h=h_0}$$

$\rightarrow \mathbf{0}$ $\qquad \rightarrow \mathbf{0}$

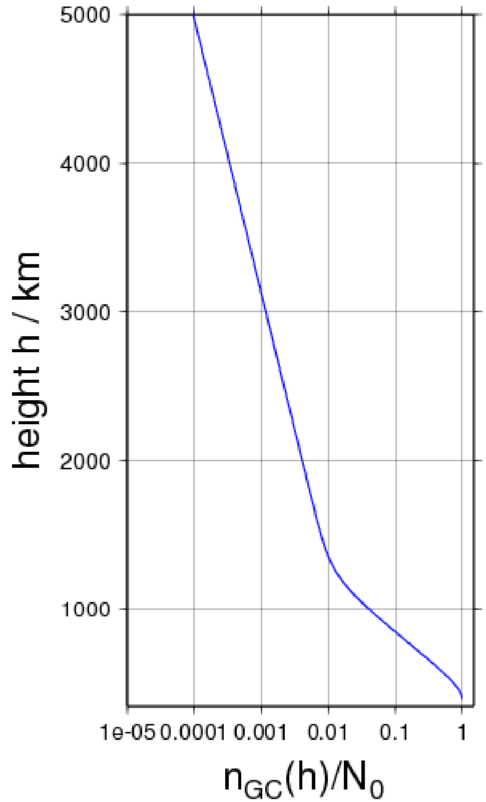
- The solution is susceptible to noise, since H depends on the derivative of n(h)

Solution for n_{GC} / 4

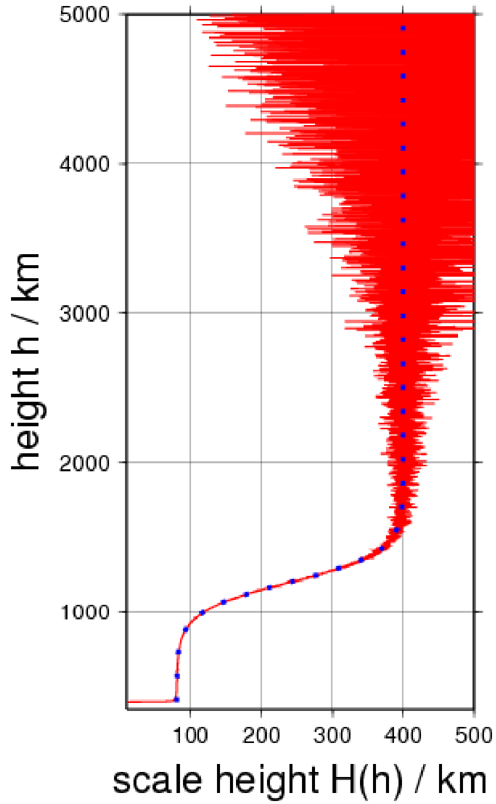
Input height function



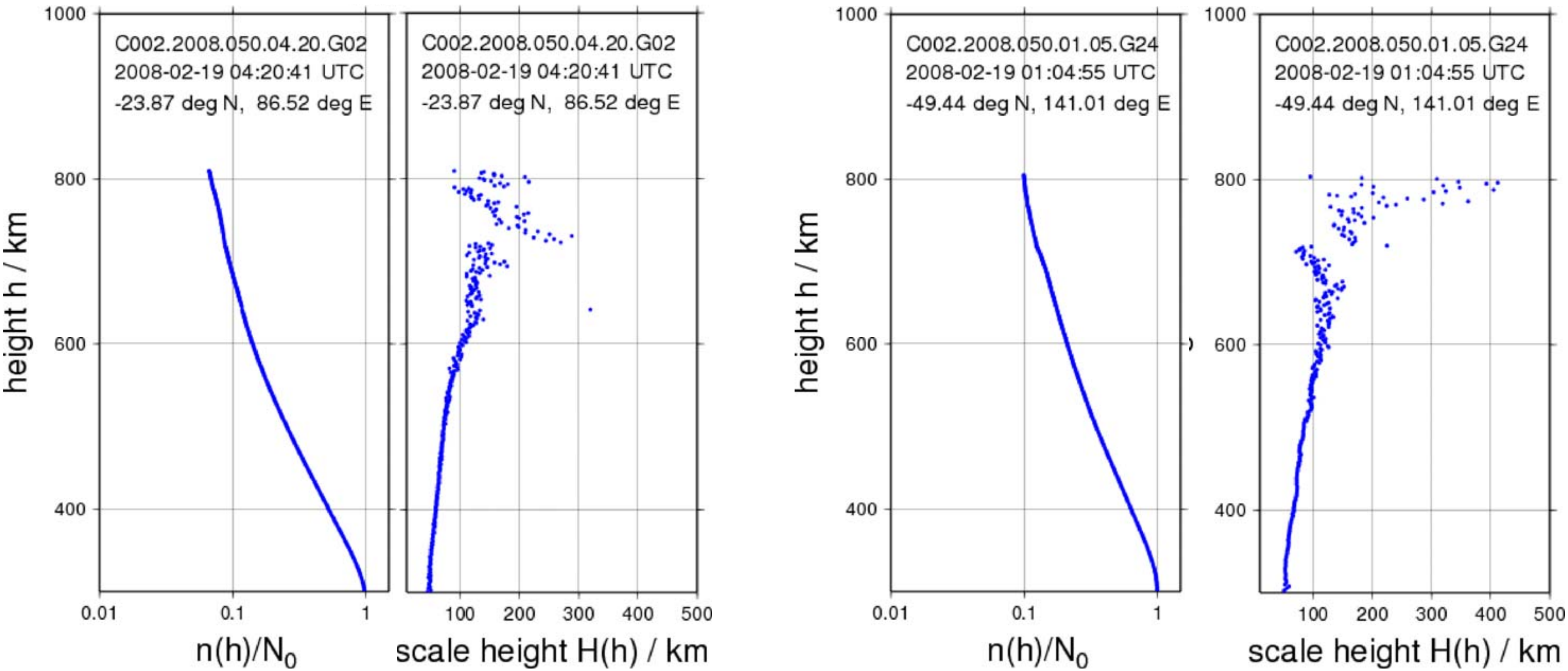
Electron density Function + 10^{-6} noise



Reconstructed height function



FORMOSAT-3/COSMIC n_{GC} reconstruction



- Input is smoothed n(h) (40 datapoint average)
- Change in scale height is clearly visible
- Noise at high altitudes

Solution for n_{VC} / 1

Solution by direct integration:

$$\tilde{n}_{VC}^{2}(h) = \frac{\mathrm{d}}{\mathrm{d}h}\exp\left(-e^{-y}\right) \qquad \tilde{n}_{VC} = \frac{n_{VC}(h)}{e\,N_0\sqrt{H_0}}$$

$$y(h) = -\ln\left\{-\ln\left(\frac{1}{e} + \int_{h_0}^{h} \tilde{n}_{VC}^{2}(h')\,\mathrm{d}h'\right)\right\}$$

- We have not yet determined N_0 and H_0
  - N_0 … maximum of electron density at $n'(h_0)=0$
  - H_0 … there is more than one way

Solution for n_{VC} / 2

Determination of H_0

1. From data near the peak electron density at $h=h_0$:

$$2\frac{n_{VC}''(h_0)}{N_0} = -H_0''(h_0) - \frac{1}{H_0^2}$$

(this is imprecise and sensitive to noise)

2. From the constraint

$$\frac{n_{VC}^2(h)}{e\,N_0^2} = -\frac{H_0}{H(h)}\cdot\left(e^{-1} + \int_{h_0}^{h}\tilde{n}_{VC}^2(h')\mathrm{d}h'\right)\cdot\ln\left(e^{-1} + \int_{h_0}^{h}\tilde{n}_{VC}^2(h')\mathrm{d}h'\right) > 0$$

we derive the bound $\int_{h_0}^{h}\tilde{n}_{VC}^2(h')\mathrm{d}h' < 1 - e^{-1}$

Solution for n_{VC} / 3

Ansatz: $$\frac{1}{H_0}\int_{h_0}^{h_\infty}\frac{n^2(h')}{N_0^2}\,\mathrm{d}h' = e-1-\varepsilon$$

ε > 0 is determined by evaluating

$$\frac{n^2(h_\infty)}{e\,N_0^2} = -\frac{H_0}{H_\infty}\cdot\left(e^{-1}+\int_{h_0}^{h_\infty}\tilde{n}_{VC}^2(h')\,\mathrm{d}h'\right)\cdot\ln\left(e^{-1}+\int_{h_0}^{h_\infty}\tilde{n}_{VC}^2(h')\,\mathrm{d}h'\right)$$

Result (to 1st order in ε): $$\varepsilon = \frac{(1-e)\cdot n_\infty^2 H_\infty}{n_\infty^2 H_\infty - \int_{h_0}^{h_\infty} n^2(h')\,\mathrm{d}h'}$$

Note: H_∞ is needed as input

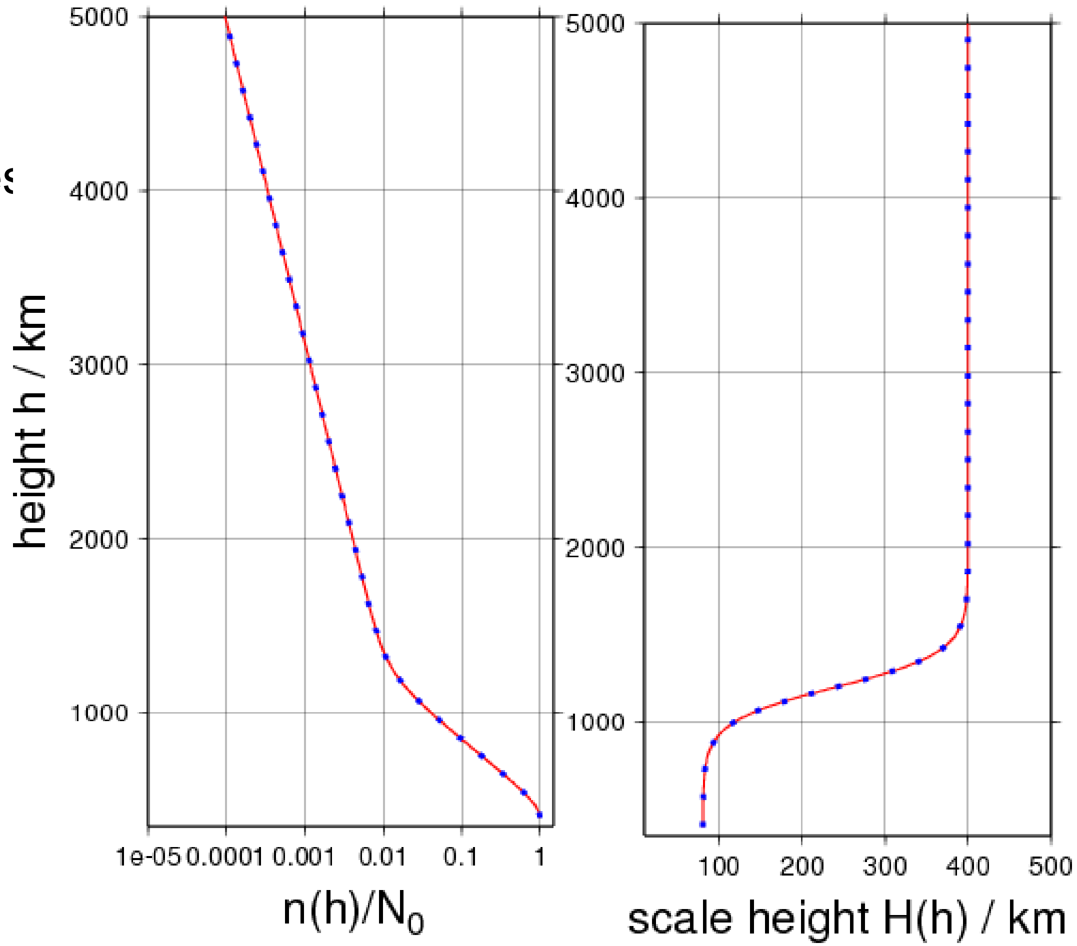
Another possible way to determine ε: $$\left.\frac{\mathrm{d}H(h)}{\mathrm{d}h}\right|_{h\to\infty} = 0$$ (future)

Solution for n_{VC} / 4

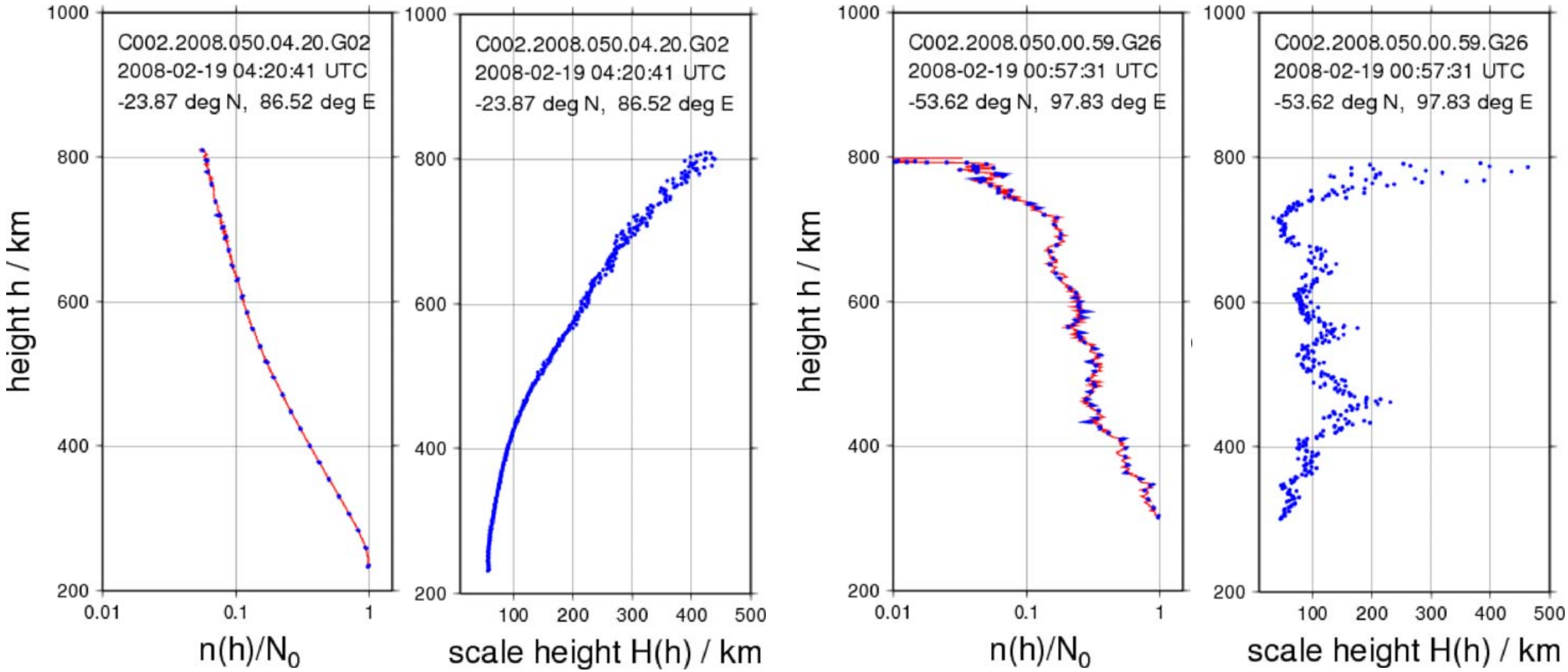
- Blue: input, red: output
- Using H_∞ = 400km
- Scale height at high altitudes is insensitive to input electron density
- Reason: *y(h)* depends on the *integral* of n_e^2
- Evaluation of this integral can reach numerical limits of type

$$1 + \varepsilon = 1$$

scale height *H(h)* reconstructed using $n_{VC}(h)$



FORMOSAT-3/COSMIC n_{VC} reconstruction



- Using H_∞ = 500km
- Reconstruction is not sensitive to noise in n(h)
- Even from irregular profiles H(h) can be reconstructed

Conclusions

- We have shown that a direct reconstruction of variable scale height functions is possible
  - for generalized Chapman layer functions n_{GC}
  - for Vary-Chap functions n_{VC}
- n_{GC}-based reconstruction:
  - inversion by Lambert W function
  - high susceptibility to noise
- n_{VC}-based reconstruction:
  - inversion by solution of a non-linear differential equation
  - determining H_0 is non-trivial, but possible