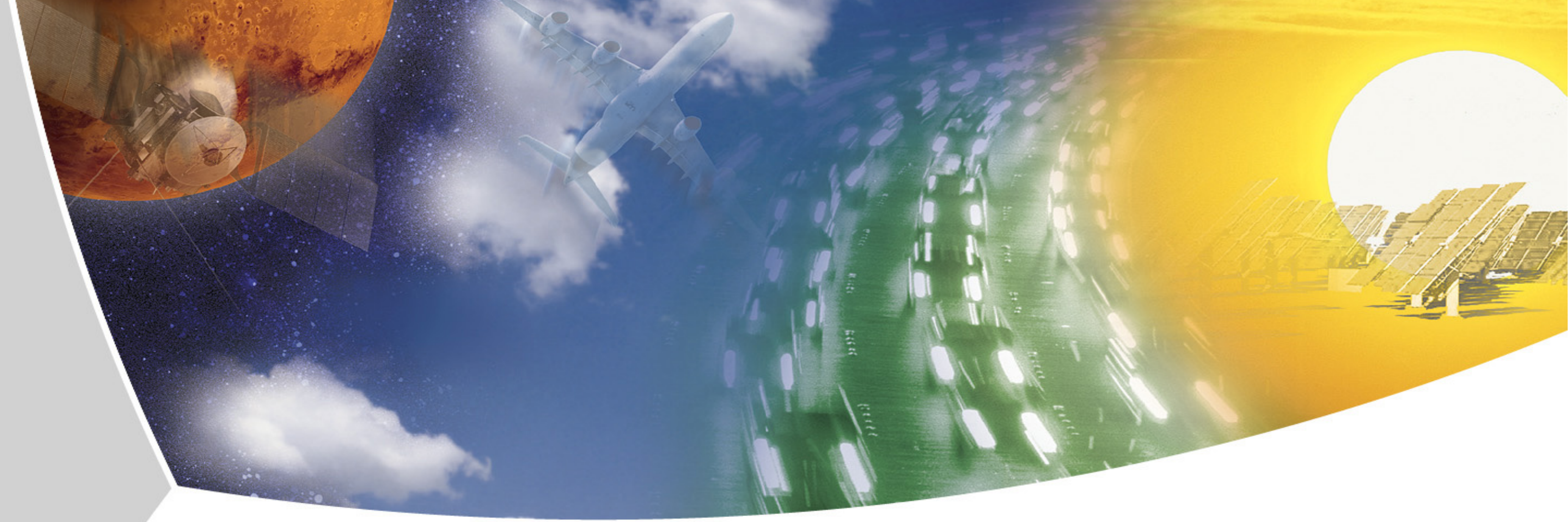

# Adaptive Discontinuous Galerkin methods in aerodynamic flow simulations

**Ralf Hartmann**

DLR Deutsches Zentrum für Luft- und Raumfahrt e.V.
in der Helmholtz-Gemeinschaft

# Adaptive DG methods in aerodynamic flow simulations

## The DG group at DLR:

- Ralf Hartmann
- Joachim Held
- Tobias Leicht
- Florian Prill

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W. Bangerth, R. Hartmann, and G. Kanschat. deal.II. A general purpose object oriented finite element library. ACM Transactions on Mathematical Software, 33(4), Aug. 2007.

W. Bangerth, R. Hartmann, and G. Kanschat. deal.II. Differential Equations Analysis Library. Technical Reference. http://www.dealii.org/, 6.1 edition, May 2008. First edition 1999.

# Problem and Discretization

## Laminar aerodynamic flow as governed by the compr. Navier-Stokes equations

- Symmetric interior penalty discontinuous Galerkin discretization
- with consistent and adjoint consistent discretization of boundary conditions
- with consistent and adjoint consistent discretization of the aerodynamic force coefficients
- with an optimal order penalty term for the compressible NS equations


R. Hartmann and P. Houston. Symmetric interior penalty DG methods for the compressible Navier–Stokes equations I: Method formulation. Int. J. Num. Anal. Model., 3(1):1–20, 2006.

R. Hartmann. Adjoint consistency analysis of discontinuous Galerkin discretizations. SIAM J. Numer. Anal., 45(6):2671–2696, 2007.

R. Hartmann and P. Houston. An optimal order interior penalty discontinuous Galerkin discretization of the compressible Navier–Stokes equations. J. Comput. Physics, 2007. In review.

# Multi target error estimation and adaptivity: Motivation

**Given $N$ target quantities, e.g. following aerodynamic force coefficients**

- **the pressure induced force coefficients: cdp, clp, cmp**
- **the viscous force coefficients: cdf, clf, cmf**

**i.e. 6 force coefficients in 2d or 10 force coefficients in 3d.**

**The standard approach: Error estimation and adjoint-based refinement requires the solution of $N$ adjoint problems.**

**Goal: Replace the $N$ adjoint problems by two auxiliary problems (1 adjoint problem and 1 adjoint adjoint problem) irrespective of the number of target quantities.**

# Error estimation for single target quantities

**Given a discretization: find $\mathrm{u}_h \in \mathrm{V}_h$ such that**

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{1}$$

**and a target quantity $J$.**
**Computed: $J(\mathrm{u}_h)$, exact (but unknown): $J(\mathrm{u})$, what is $J(\mathrm{u}) - J(\mathrm{u}_h)$?!**

**Using a duality argument we obtain an error representation wrt. $J(\cdot)$:**

$$\begin{aligned} J(\mathbf{u}) - J(\mathbf{u}_h) &= \mathcal{R}(\mathbf{u}_h, \mathbf{z}) := -\mathcal{N}(\mathbf{u}_h, \mathbf{z}) \\ &\approx \mathcal{R}(\mathbf{u}_h, \tilde{\mathbf{z}}_h) = \sum_{\kappa} \eta_\kappa. \end{aligned}$$

**where $\tilde{\mathrm{z}}_h$ is the solution to the discrete adjoint problem: find $\tilde{\mathrm{z}}_h \in \tilde{\mathrm{V}}_h$ such that**

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}_h, \tilde{\mathbf{z}}_h) = J'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h,$$

**and $\eta_\kappa$ are adjoint-based indicators which are particularly suited for the accurate and efficient approximation of the target quantity $J(\mathrm{u})$.**

# Single target error estimation and adaptivity

## applied to Discontinuous Galerkin discretizations of

- the linear advection equation
- scalar nonlinear conservation laws:
  inviscid Burgers equation, Buckley-Leverett equation
- 1d compressible Euler equations
- 2d compressible Euler equations: subsonic, transonic and supersonic flows
- 2d compressible Navier-Stokes equations: subsonic, transonic, supersonic flows

# Single target error estimation and adaptivity

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R. Hartmann. Adaptive Finite Element Methods for the Compressible Euler Equations. PhD thesis, University of Heidelberg, 2002.

R. Hartmann and P. Houston. Adaptive discontinuous Galerkin finite element methods for nonlinear hyperbolic conservation laws. SIAM J. Sci. Comp., 24:979–1004, 2002.

R. Hartmann and P. Houston. Adaptive discontinuous Galerkin finite element methods for the compressible Euler equations. J. Comput. Phys., 183:508–532, 2002.

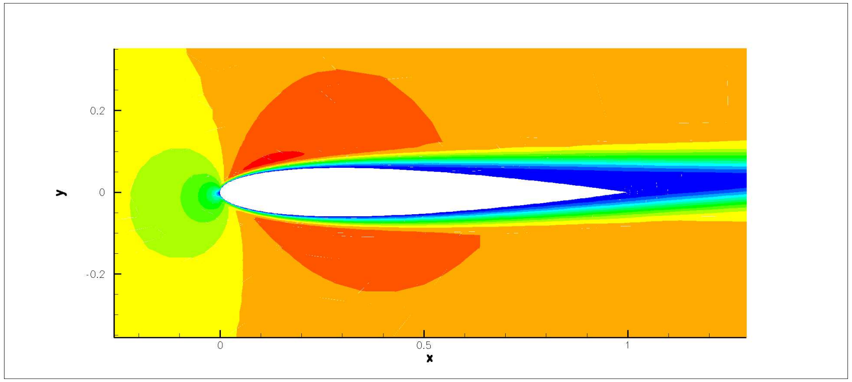
R. Hartmann and P. Houston. Symmetric interior penalty DG methods for the compressible Navier–Stokes equations II: Goal–oriented a posteriori error estimation. Int. J. Num. Anal. Model., 3(2):141–162, 2006.

R. Hartmann. Adaptive discontinuous Galerkin methods with shock-capturing for the compressible Navier-Stokes equations. Int. J. Numer. Meth. Fluids, 51(9–10):1131–1156, 2006.

T. Leicht and R. Hartmann. Anisotropic mesh refinement for discontinuous Galerkin methods in two-dimensional aerodynamic flow simulations. Int. J. Numer. Meth. Fluids, 56(11):2111–2138, April 2008.

# Example: ADIGMA MTC3 test case

**Laminar flow, $M = 0.5, \alpha = 2°, Re = 5000$ around NACA0012 airfoil:**




**We are interested in the**

1. **pressure induced drag:** $J(\mathrm{u}) = c_{dp}$
2. **viscous drag:** $J(\mathrm{u}) = c_{df}$
3. **total lift:** $J(\mathrm{u}) = c_l$
4. **total momentum:** $J(\mathrm{u}) = c_m$

# Error estimation for single target quantity: $J(\mathbf{u}) = c_{dp}$

**Example: MTC-3, laminar flow,** $M = 0.5, \alpha = 2^\circ, Re = 5000$

**Target quantity:** $J(\mathbf{u}) = c_{dp}$ **(pressure induced drag), Ref.value:** $J_{cdp}^{ref}(\mathbf{u}) = 0.02380$

| | | error in $c_{dp}$ | | |
|---|---|---|---|---|
| cells | DoFs | exact | estimate | ratio |
| 400 | 6400 | 1.034e-03 | -1.404e-03 | -1.36 |
| 652 | 10432 | 3.341e-03 | 2.959e-03 | 0.89 |
| 1090 | 17440 | 4.045e-04 | 5.712e-04 | 1.41 |
| 1801 | 28816 | -2.079e-04 | -1.091e-04 | 0.52 |
| 3034 | 48544 | -2.344e-04 | -1.890e-04 | 0.81 |
| 5047 | 80752 | -1.529e-04 | -1.387e-04 | 0.91 |
| 8527 | 136432 | -8.055e-05 | -7.536e-05 | 0.94 |
| 14410 | 230560 | -4.357e-05 | -3.762e-05 | 0.86 |
| 24406 | 390496 | -2.366e-05 | -2.314e-05 | 0.98 |

# Error estimation for single target quantity: $J(\mathrm{u}) = c_{df}$

**Example: MTC-3, laminar flow,** $M = 0.5, \alpha = 2^\circ, Re = 5000$

**Target quantity:** $J(\mathrm{u}) = c_{df}$ **(viscous drag), Ref.value:** $J^{ref}_{cdf}(\mathrm{u}) = 0.0322835$

| | | error in $c_{df}$ | | |
|---|---|---|---|---|
| **cells** | **DoFs** | **exact** | **estimate** | **ratio** |
| 400 | 6400 | 1.076e-02 | 1.525e-02 | 1.42 |
| 655 | 10480 | -2.973e-03 | -2.592e-03 | 0.87 |
| 1093 | 17488 | -1.415e-03 | -1.418e-03 | 1.00 |
| 1804 | 28864 | -3.947e-04 | -4.326e-04 | 1.10 |
| 2989 | 47824 | -9.136e-05 | -1.116e-04 | 1.22 |
| 5110 | 81760 | -3.787e-05 | -4.518e-05 | 1.19 |
| 8476 | 135616 | -1.919e-05 | -2.071e-05 | 1.08 |
| 14185 | 226960 | -1.319e-05 | -1.619e-05 | 1.23 |
| 23638 | 378208 | -1.048e-05 | -1.052e-05 | 1.00 |

# Error estimation for single target quantity: $J(\mathbf{u}) = c_l$

**Example: MTC-3, laminar flow,** $M = 0.5, \alpha = 2^\circ, Re = 5000$

**Target quantity:** $J(\mathbf{u}) = c_l$ **(total lift), Ref.value:** $J_{cl}^{ref}(\mathbf{u}) = 0.037286$

| | | error in $c_l$ | | |
|---|---|---|---|---|
| **cells** | **DoFs** | **exact** | **estimate** | **ratio** |
| 400 | 6400 | -1.175e-01 | -5.867e-02 | 0.50 |
| 658 | 10528 | 6.548e-03 | 6.841e-03 | 1.04 |
| 1108 | 17728 | -1.292e-03 | -1.159e-03 | 0.90 |
| 1861 | 29776 | -1.784e-03 | -1.891e-03 | 1.06 |
| 3118 | 49888 | -1.239e-03 | -1.266e-03 | 1.02 |
| 5236 | 83776 | -6.504e-04 | -6.704e-04 | 1.03 |
| 8746 | 139936 | -2.623e-04 | -2.622e-04 | 1.00 |

# Error estimation for single target quantity: $J(\mathbf{u}) = c_m$

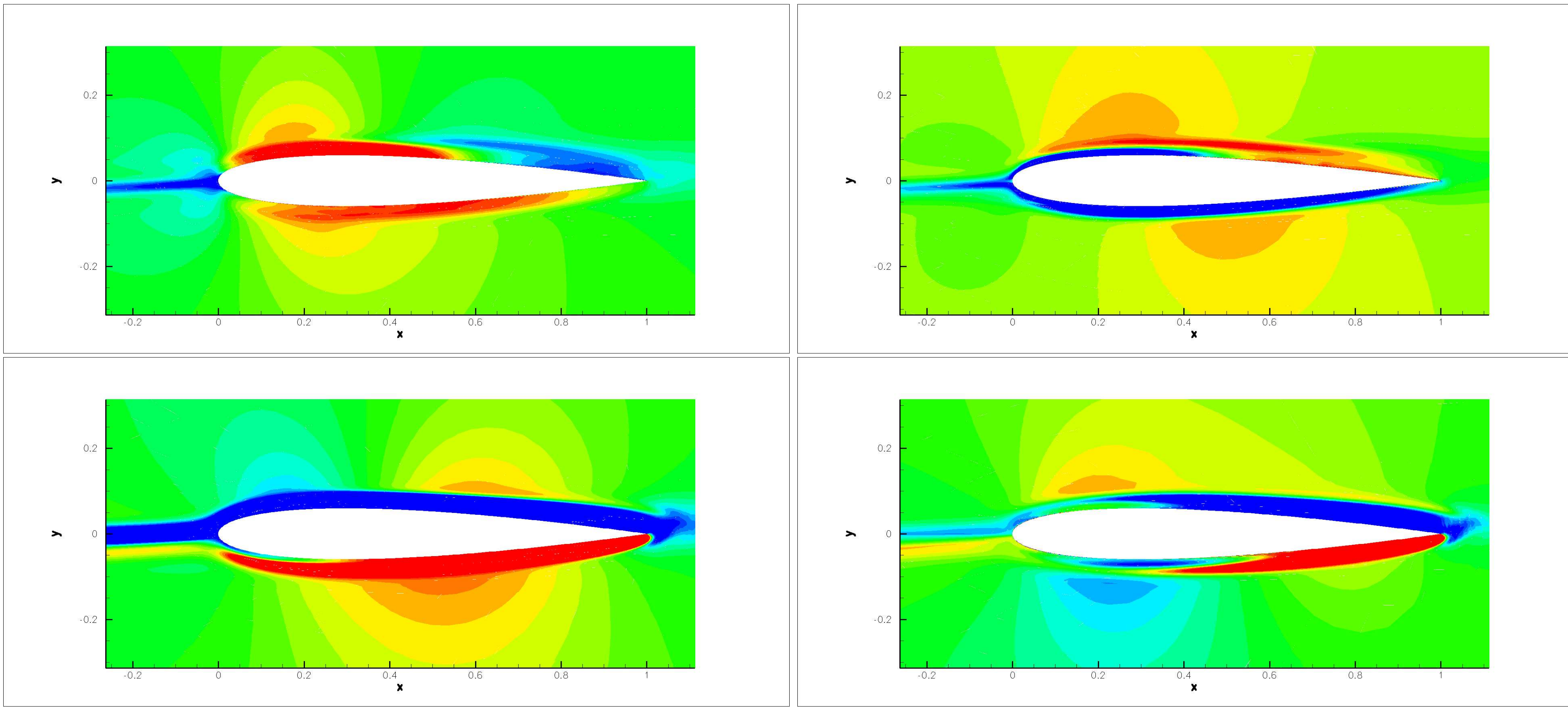
**Example: MTC-3, laminar flow,** $M = 0.5, \alpha = 2^\circ, Re = 5000$

**Target quantity:** $J(\mathbf{u}) = c_m$ **(total moment), Ref.value:** $J_{cm}^{ref}(\mathbf{u}) = -0.01661$

| | | error in $c_m$ | | |
|---|---|---|---|---|
| cells | DoFs | exact | estimate | ratio |
| 400 | 6400 | -1.221e-03 | -3.035e-03 | 2.49 |
| 667 | 10672 | 2.883e-03 | 3.001e-03 | 1.04 |
| 1138 | 18208 | 3.862e-04 | 4.378e-04 | 1.13 |
| 1867 | 29872 | 9.083e-05 | 8.543e-05 | 0.94 |
| 3130 | 50080 | 6.199e-05 | 5.807e-05 | 0.94 |

# Error estimation for single target quantities

$z_1$ **components of adjoint solutions.** **Top: cdp, cdf;** **Bottom: cl, cm.**



# Error estimation for multiple target quantities

**For $N$ target quantities $J_i(\mathbf{u}), i = 1, \ldots, N$: Instead of computing $N$ adjoint solutions**

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}_h, \tilde{\mathbf{z}}_{i,h}) = J_i'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h, \qquad i = 1, \ldots, N,$$

**to obtain error estimates for the $N$ target quantities**

$$J_i(\mathbf{u}) - J_i(\mathbf{u}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{z}_i) \approx \mathcal{R}(\mathbf{u}_h, \mathbf{z}_{i,h}), \qquad i = 1, \ldots, N,$$

**we now solve *one* discrete error equation: find $\tilde{\mathrm{e}}_h \in \tilde{\mathbf{V}}_h$ such that**

$$\mathcal{N}'[\mathbf{u}_h](\tilde{\mathbf{e}}_h, \mathbf{w}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h,$$

**to obtain error estimates for the $N$ target quantities**

$$J_i(\mathbf{u}) - J_i(\mathbf{u}_h) \approx J_i'[\mathbf{u}_h](\mathbf{e}) \approx J_i'[\mathbf{u}_h](\tilde{\mathbf{e}}_h), \qquad i = 1, \ldots, N.$$

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R. Hartmann and P. Houston. Goal-oriented a posteriori error estimation for multiple target functionals. In T. Y. Hou and E. Tadmor, eds, Hyperbolic problems: theory, numerics, applications, pp. 579–588. Springer, 2003.

R. Hartmann. Multi-target error estimation and adaptivity in aerodynamic flow simulations. SIAM J. Sci. Comput., 2008. To appear.

# Error estimation for multiple target quantities

**Example: MTC-3, laminar flow, $M = 0.5, \alpha = 2^\circ, Re = 5000$**

**On each mesh compute primal solution $\mathrm{u}_h$ and adjoint-adjoint solution $\tilde{\mathrm{e}}_h$.**

**Evaluate *exact* error:** $J_i^{\mathrm{ref}}(\mathbf{u}) - J_i(\mathbf{u}_h), \quad i = 1, \ldots, N,$

**Evaluate error *estimate*:** $J_i'[\mathbf{u}_h](\tilde{\mathrm{e}}_h), \quad i = 1, \ldots, N,$

| | error in cdp | | error in cdf | | error in cl | | error in cm | |
|---|---|---|---|---|---|---|---|---|
| #cells | exact | estimate | exact | estimate | exact | estimate | exact | estimate |
| 400 | 1.03e-03 | -2.92e-03 | 1.08e-02 | 1.62e-02 | -1.18e-01 | -6.59e-02 | -1.22e-03 | -4.36e-03 |
| 655 | 1.39e-03 | 1.38e-03 | -3.02e-03 | -2.89e-03 | 6.30e-03 | 4.15e-03 | 2.99e-03 | 2.67e-03 |
| 1111 | -1.04e-04 | 8.65e-05 | -1.42e-03 | -1.89e-03 | -8.30e-04 | -6.54e-04 | 4.76e-04 | 5.11e-04 |
| 1843 | -6.28e-04 | -5.28e-04 | -5.20e-04 | -6.46e-04 | -1.83e-03 | -1.91e-03 | 6.25e-05 | 3.49e-05 |
| 3061 | -3.96e-04 | -3.51e-04 | -1.61e-04 | -2.25e-04 | -7.34e-04 | -7.69e-04 | 3.15e-05 | 3.67e-05 |
| 5146 | -1.82e-04 | -1.63e-04 | -9.03e-05 | -1.11e-04 | -4.86e-04 | -3.94e-04 | 1.09e-05 | 1.35e-05 |

# Adaptive refinement for multiple target functionals (1)

**Goal-oriented mesh refinement tailored to reducing e.g.**

**a) the sum of relative errors or b) the (weighted) sum of absolute errors:**

$$a) \quad \sum_{i=1}^{N} |J_i(\mathbf{u}) - J_i(\mathbf{u}_h)|/|J_i(\mathbf{u})|, \qquad b) \quad \sum_{i=1}^{N} \alpha_i |J_i(\mathbf{u}) - J_i(\mathbf{u}_h)|.$$

**Define the combined target functional:**

$$a) \quad J_c(\mathbf{v}) = \sum_{i=1}^{N} s_i J_i(\mathbf{v})/|J_i(\mathbf{u}_h)|, \qquad b) \quad J_c(\mathbf{v}) = \sum_{i=1}^{N} \alpha_i s_i J_i(\mathbf{v}),$$

**with** $s_i =$ **sign**$(J_i(\mathbf{u}) - J_i(\mathbf{u}_h))$**.**

**We now solve the adjoint problem: find** $\tilde{\mathbf{z}}_{c,h} \in \tilde{\mathbf{V}}_h$ **such that**

$$\mathcal{N}'[\mathbf{u}_h](\mathbf{w}_h, \tilde{\mathbf{z}}_{c,h}) = J_c'[\mathbf{u}_h](\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \tilde{\mathbf{V}}_h, \qquad i = 1, \ldots, N,$$

**and obtain the error estimate**

$$J_c(\mathbf{u}) - J_c(\mathbf{u}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{z}_c) \approx \mathcal{R}(\mathbf{u}_h, \tilde{\mathbf{z}}_{c,h}) = \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa.$$

# Adaptive refinement for multiple target functionals (2)

**Example: MTC-3, laminar flow,** $M = 0.5, \alpha = 2^\circ, Re = 5000$

**Target quantity: sum of relative errors**

| # cells | # DoFs | exact | estimate | ratio |
|---|---|---|---|---|
| 400 | 6400 | 3.602e+00 | 1.362e+00 | 0.38 |
| 655 | 10480 | 5.009e-01 | 5.036e-01 | 1.01 |
| 1111 | 17776 | 9.940e-02 | 9.135e-02 | 0.92 |
| 1843 | 29488 | 9.535e-02 | 8.884e-02 | 0.93 |
| 3061 | 48976 | 4.320e-02 | 4.299e-02 | 1.00 |
| 5146 | 82336 | 2.414e-02 | 2.537e-02 | 1.05 |

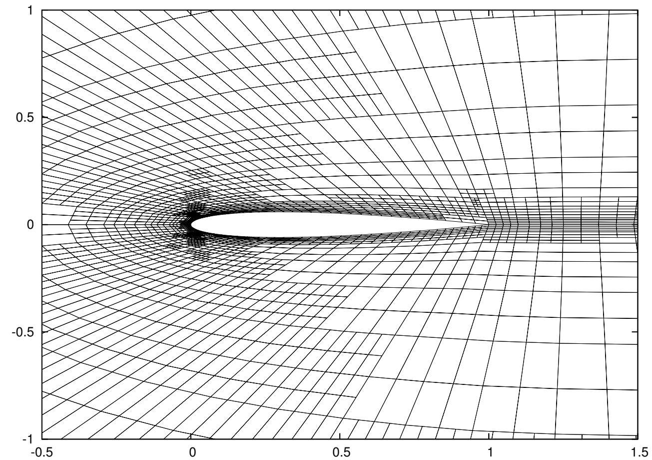
**On finest mesh sum of relative errors is 2.4%. Error estimation tells us: 2.5%**

## Goal-oriented refinement for multiple target quantities

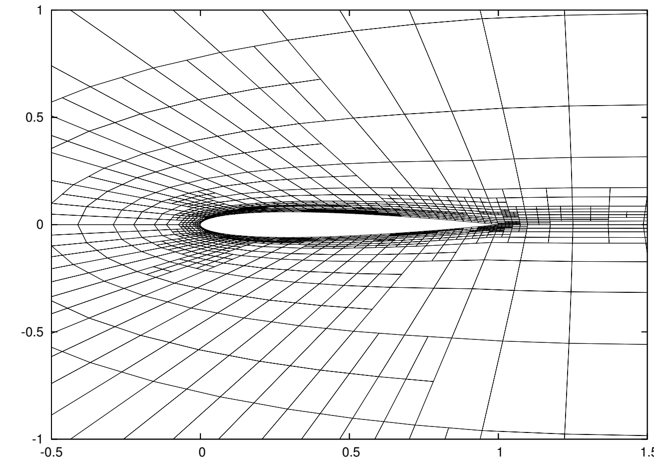
**Example: ADIGMA MTC-3, laminar, $M = 0.5, \alpha = 2^\circ, Re = 5000$**

**Goal: Accurate and efficient approximation of cdp, cdf, cl, cm**

**accuracy requirements (ADIGMA): cdp, cdf, cm: |error| <5e-4, cl: |error| <5e-3**




**residual-based refinement**
**8896 cells, 149.4s**




**adjoint-based refinement**
**1894 cells, 80.8s (incl. error est.)**

**stronger accuracy requirements: cdp, cdf, cm: |error| <1e-4, cl: |error| <1e-3**

**67660 cells, 2691.1s**

**8539 cells, 664.6s (incl. error est.)**

# The residual-based indicators

**Using the error representation**

$$J(\mathbf{u}) - J(\mathbf{u}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{z}) = -\mathcal{N}(\mathbf{u}_h, \mathbf{z}) = -\mathcal{N}(\mathbf{u}_h, \mathbf{z} - \mathbf{z}_h)$$

**and assuming $\mathbf{z} \in [H^1(\kappa)]^5$ with $\|\mathbf{z}\|_{[H^1(\kappa)]^5} \leq C_{stab}$ we obtain**

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \leq \left( \sum_{\kappa \in \mathcal{T}_h} \left( \eta_\kappa^{(\mathrm{res})} \right)^2 \right)^{1/2},$$

**where the residual-based indicators $\eta_\kappa^{(\mathrm{res})}$, $\kappa \in \mathcal{T}_h$, are given by**

$$\eta_\kappa^{(\mathrm{res})} = h_\kappa \|\mathbf{R}(\mathbf{u}_h)\|_\kappa + h_\kappa^{1/2} \|\mathbf{r}_{\partial\kappa}(\mathbf{u}_h)\|_{\partial\kappa} + h_\kappa^{-1/2} \|\underline{\rho}_{\partial\kappa}(\mathbf{u}_h)\|_{\partial\kappa},$$

**Note: $\eta_\kappa^{(\mathrm{res})}$ is independent of target quantity $J(\mathbf{u})$. Mesh refinement based on the residual-based indicators $\eta_\kappa^{(\mathrm{res})}$ targets at resolving *all* flow features.**

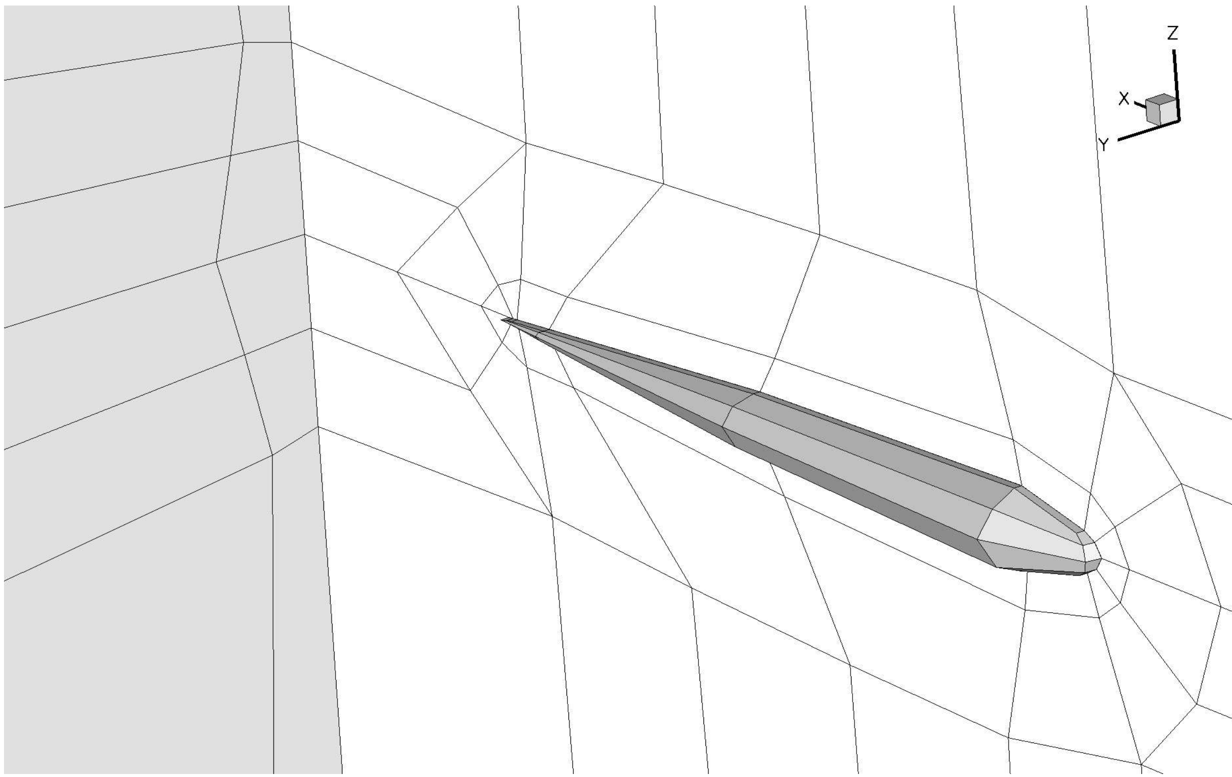
# Interface to CAD data

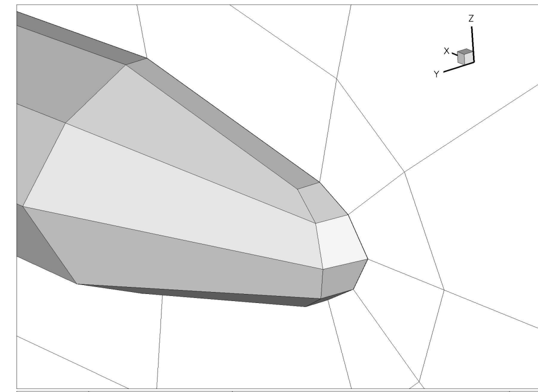
We use an interface to CAD data (via OpenCascade). It provides additional points on curved boundaries represented by CAD data. They are required

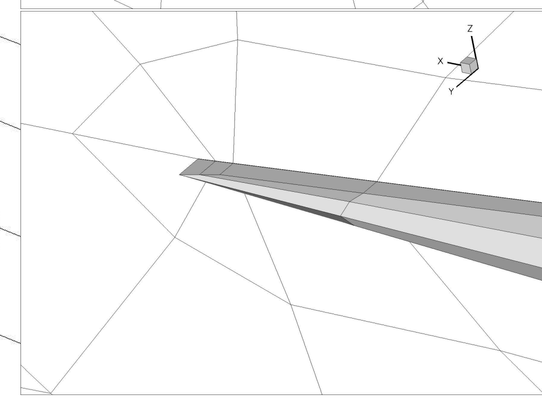
- to allow a higher order approximation of curved boundaries
- to make sure that local mesh refinement fits the CAD boundary

# Residual-based refinement for a streamlined body (BTC0)

**Freestream conditions:** $M = 0.5$, $\alpha = 1^\circ$, $Re = 5000$.



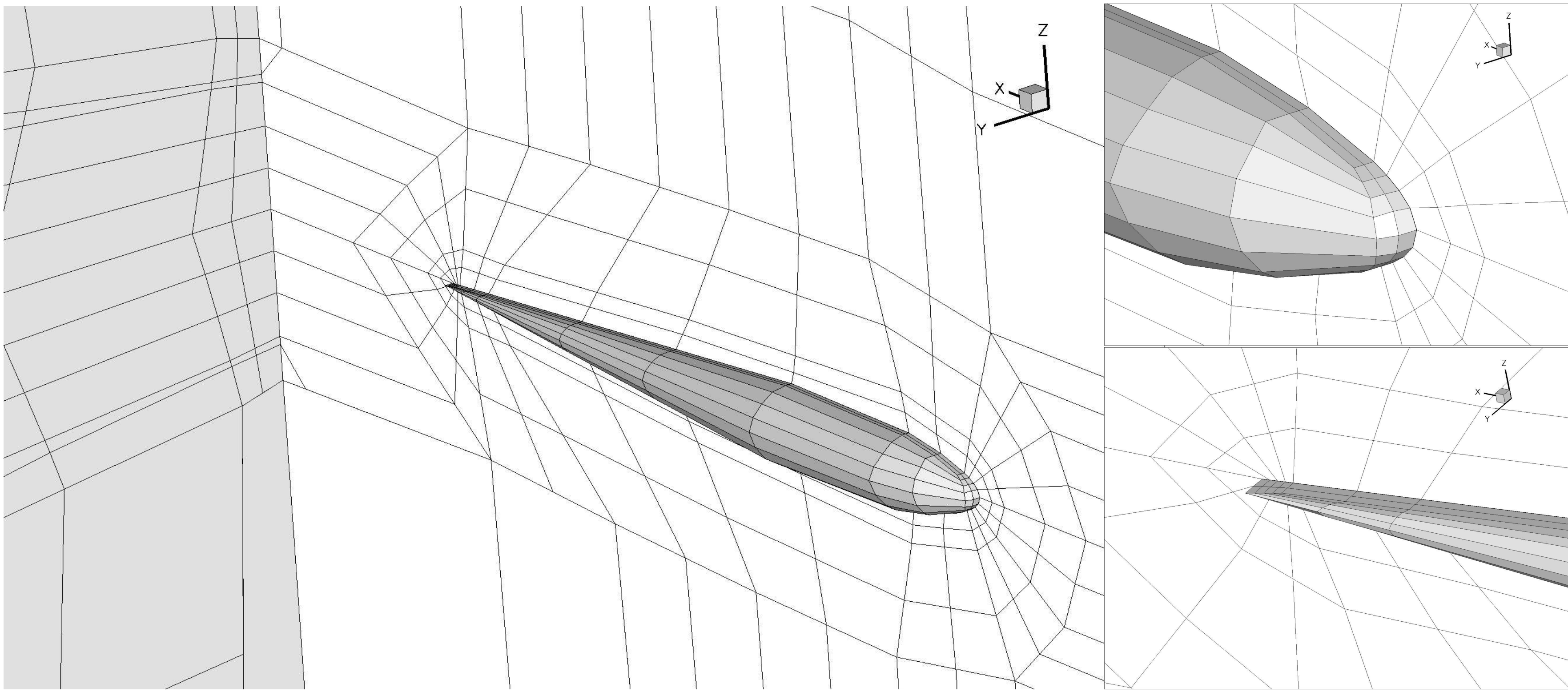





**Coarse mesh: 768 elements, 30 720 DoFs**

# Residual-based refinement for a streamlined body (BTC0)

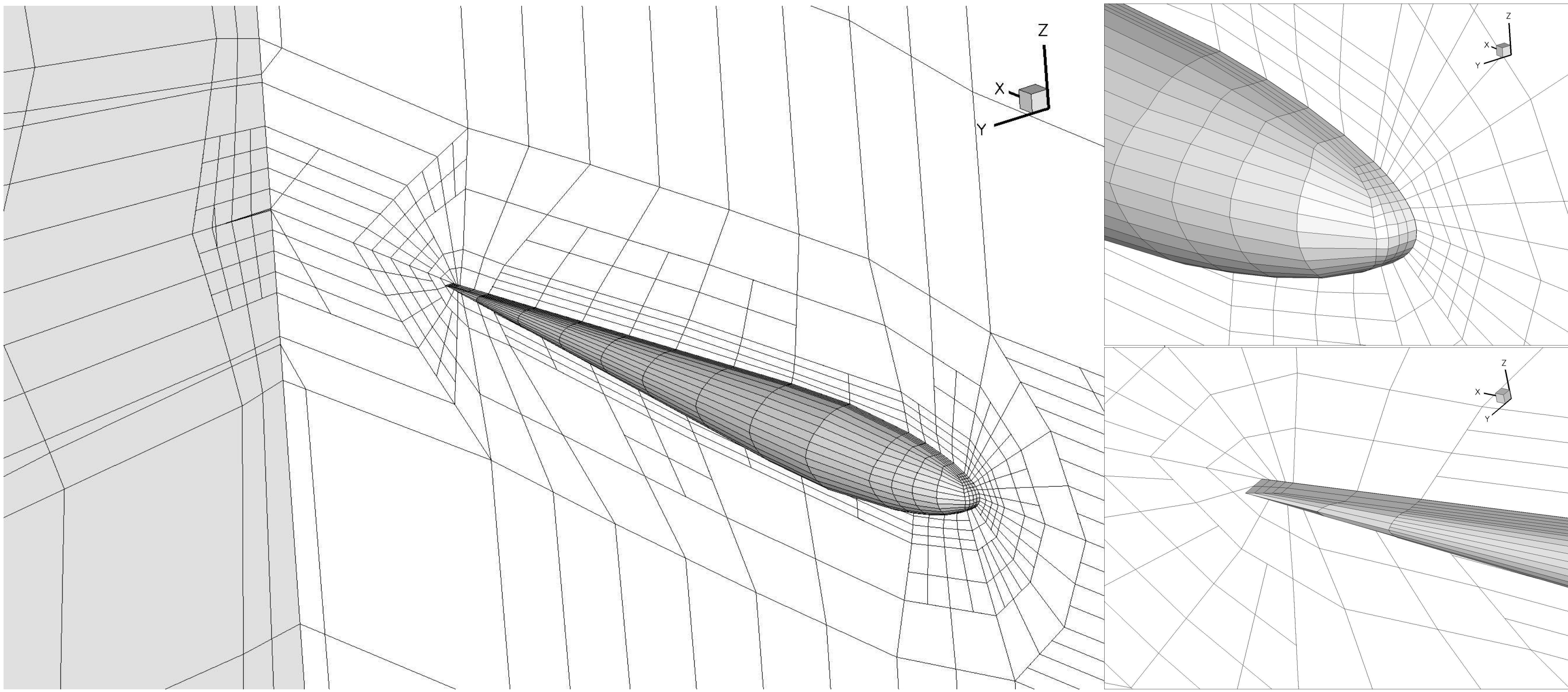
**Freestream conditions:** $M = 0.5$, $\alpha = 1^\circ$, $Re = 5000$.




**1. refinement step: 1 909 elements, 76 360 DoFs**

# Residual-based refinement for a streamlined body (BTC0)

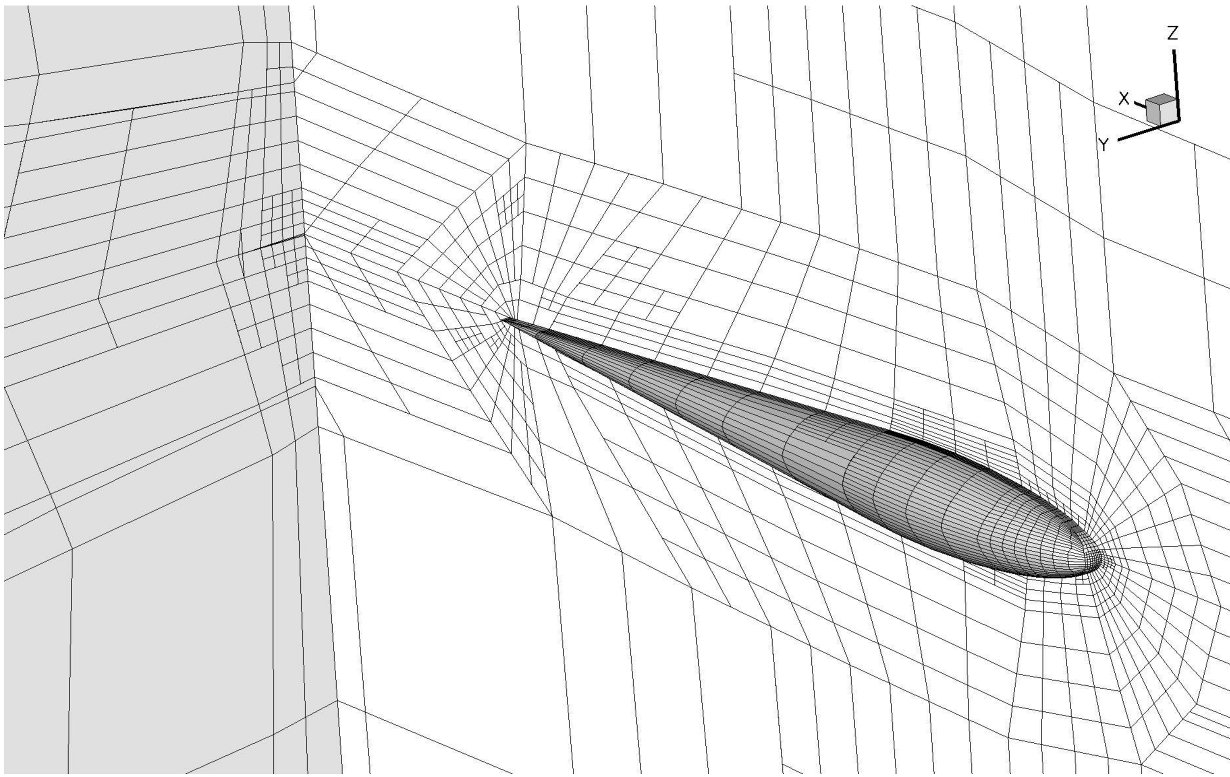
**Freestream conditions:** $M = 0.5$, $\alpha = 1^\circ$, $Re = 5000$.

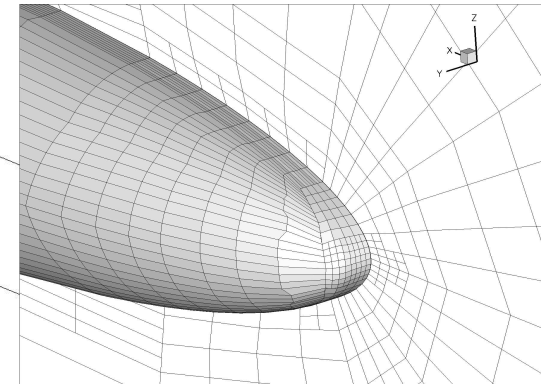


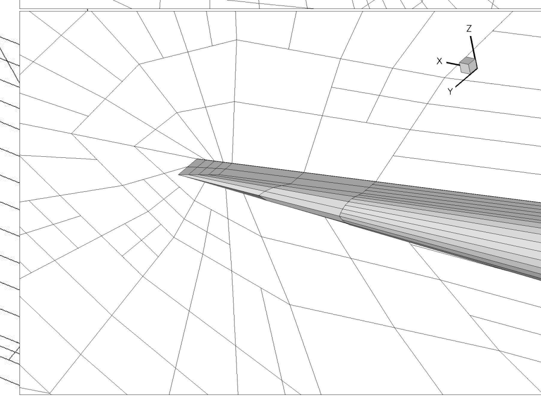

**2. refinement step: 4 912 elements, 196 480 DoFs**

# Residual-based refinement for a streamlined body (BTC0)

**Freestream conditions:** $M = 0.5$, $\alpha = 1^\circ$, $Re = 5000$.



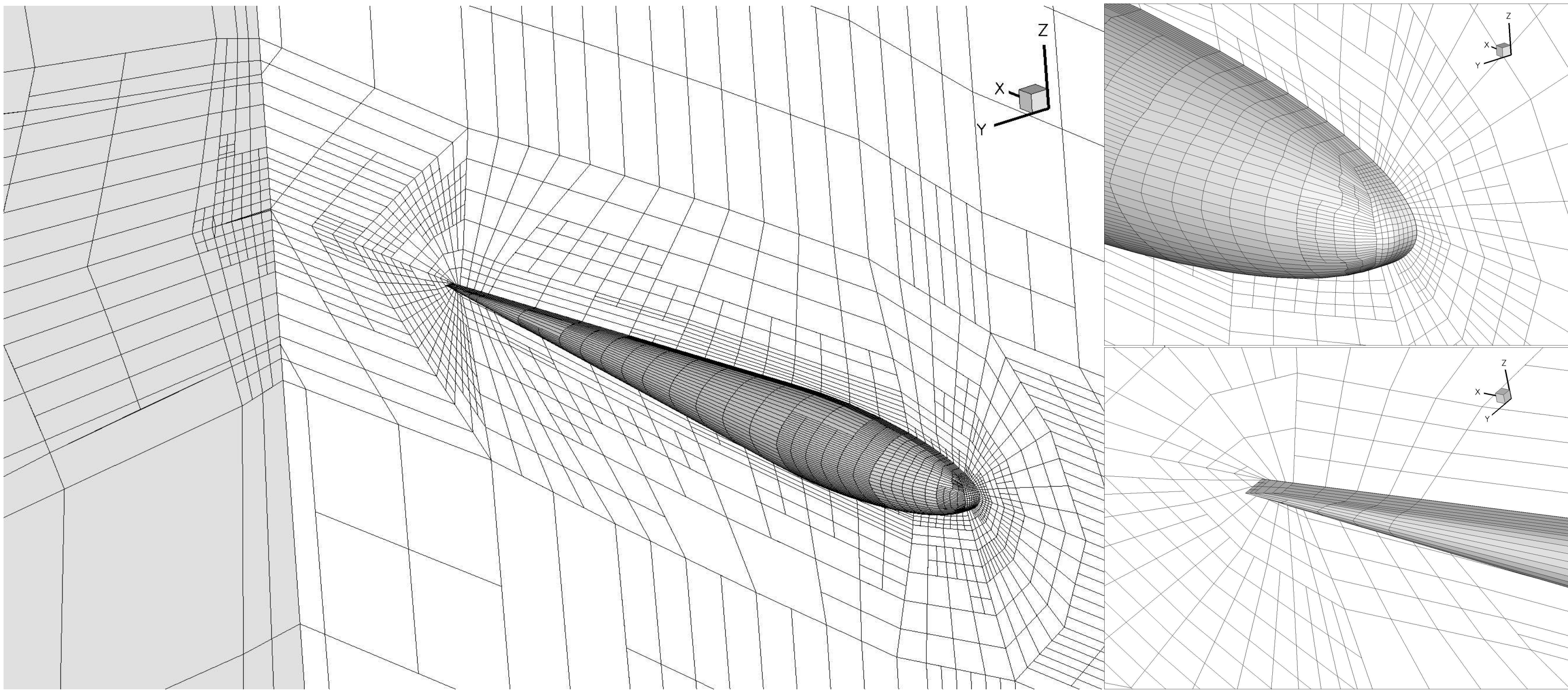





**3. refinement step: 12 437 elements, 497 480 DoFs**

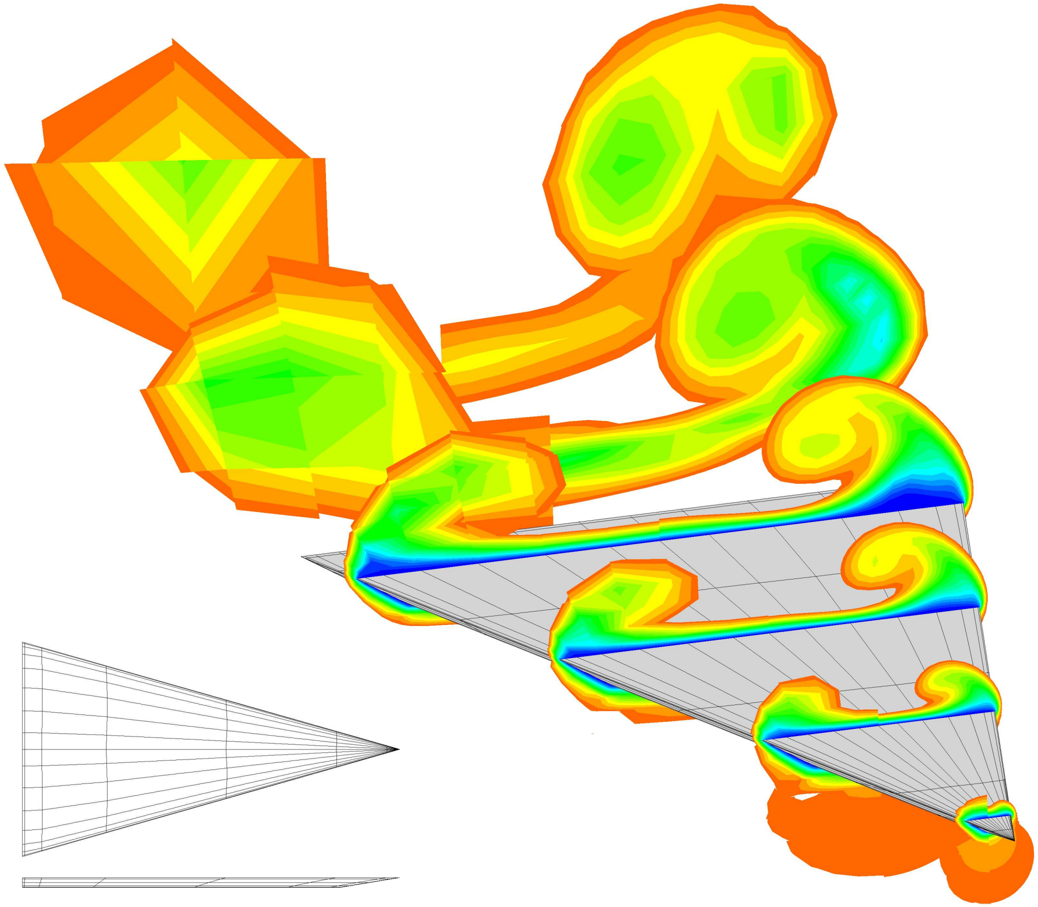
# Residual-based refinement for a streamlined body (BTC0)

**Freestream conditions:** $M = 0.5$, $\alpha = 1^\circ$, $Re = 5000$.




**4. refinement step: 31 582 elements, 1 263 280 DoFs**

# Example: Laminar delta wing (BTC3)

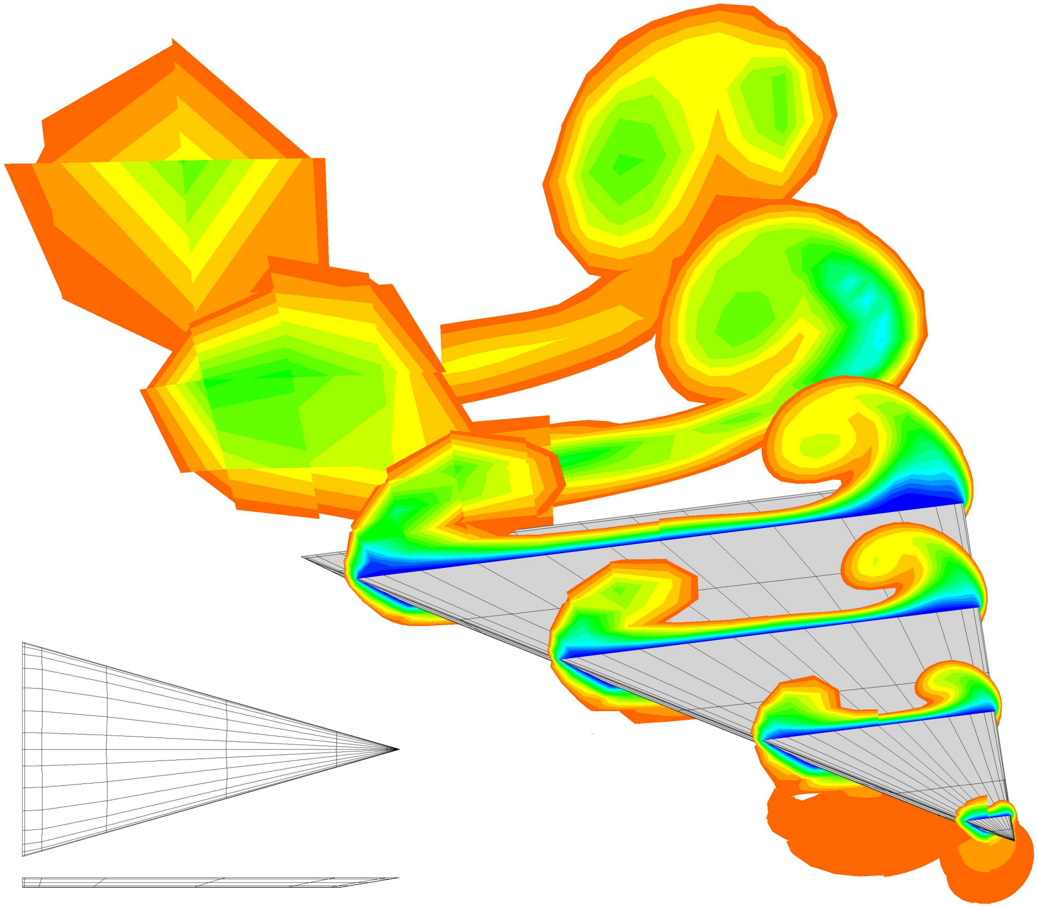


$M = 0.3, \alpha = 12.5°$,
$Re = 4000$

**3264 elements for the half model**

**left: DG(1), 2nd order**
**right: DG(4), 5th order**

# Example: Laminar delta wing (BTC3)



$M = 0.3, \alpha = 12.5°$,
$Re = 4000$

**3264 elements for the half model**

**left: DG(1), 2nd order**
**right: DG(4), 5th order**

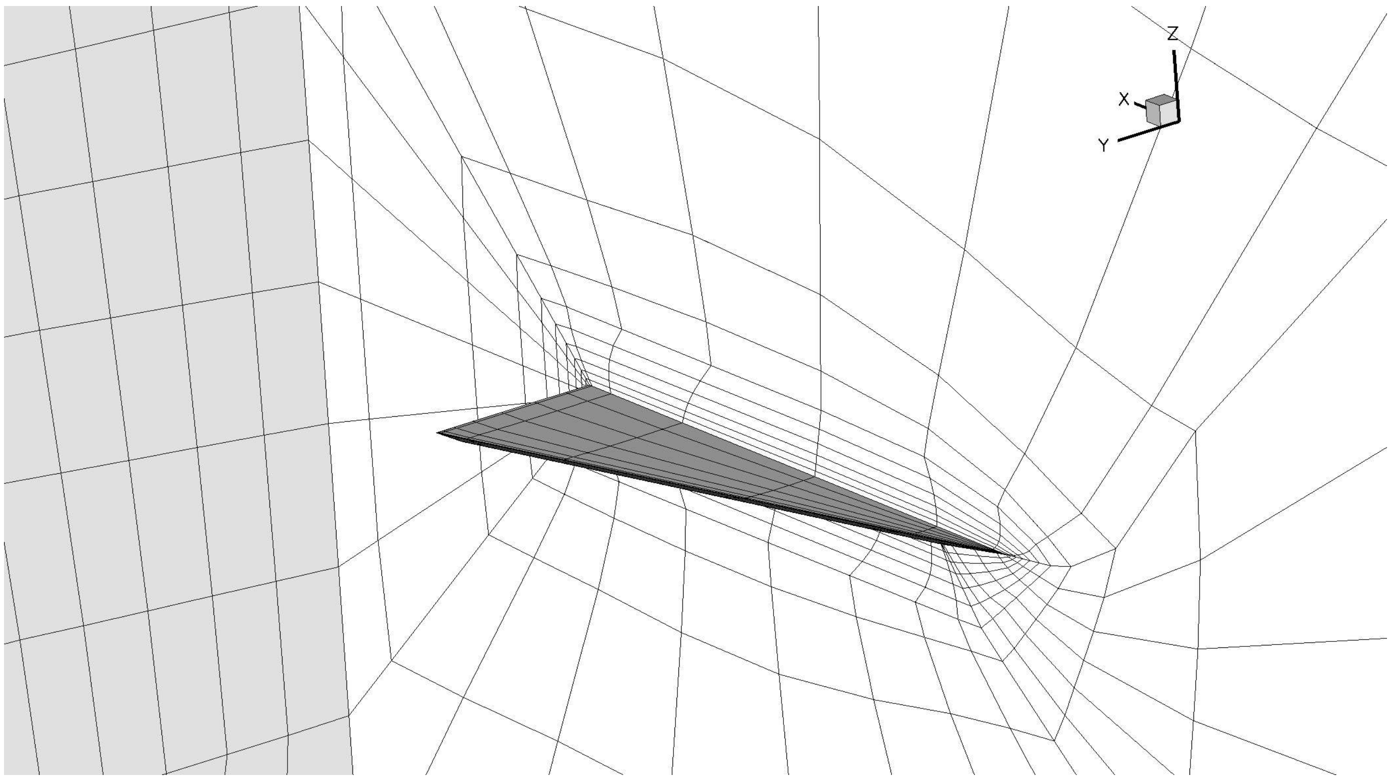
**DG(1), 40 DoFs/element:**
**130 560 DoFs**

**DG(4), 625 DoFs/element:**
**2 040 000 DoFs**

# Residual-based refinement for the laminar delta wing (BTC3)

**Freestream conditions:** $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$.

**Residual-based refinement**

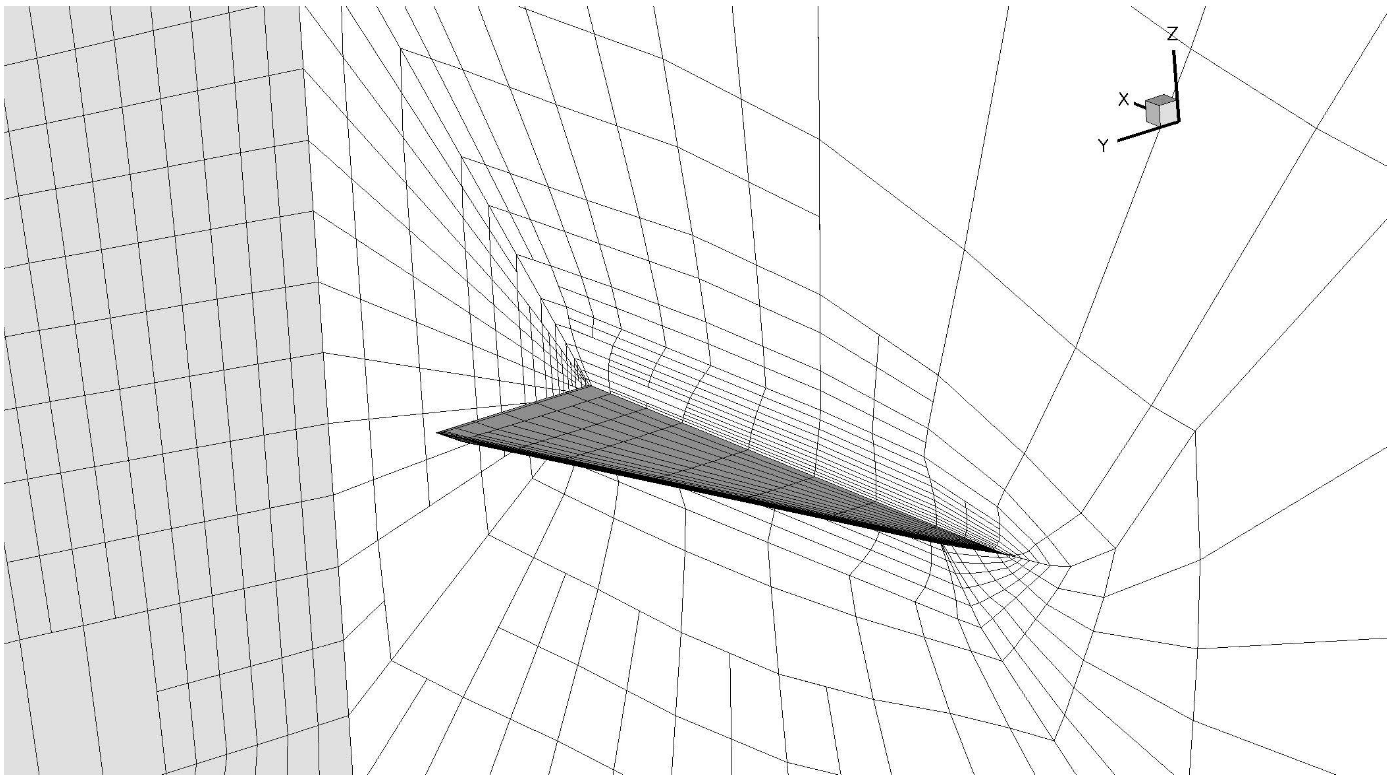



**Coarse mesh: 3 264 elements, 130 560 DoFs**

# Residual-based refinement for the laminar delta wing (BTC3)

**Freestream conditions:** $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$.

**Residual-based refinement**

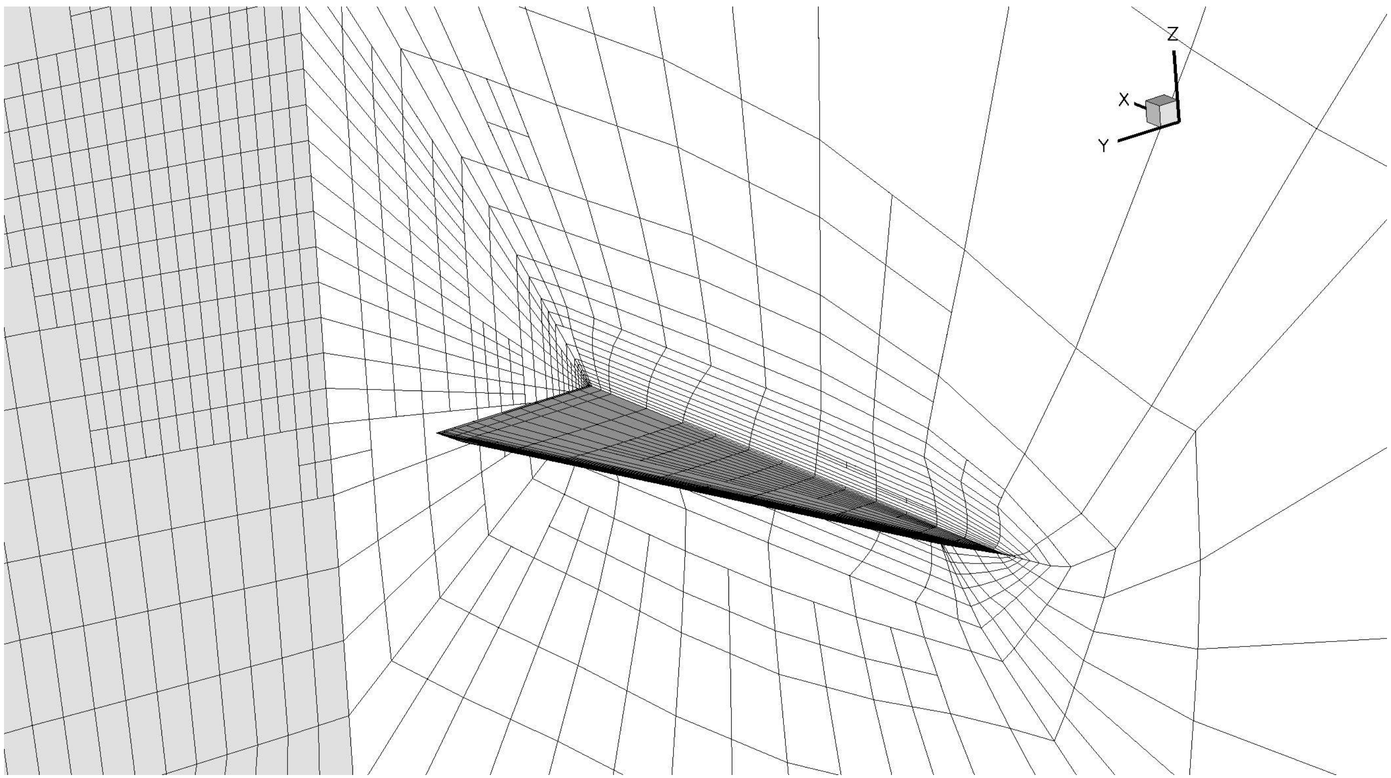



**1. refinement step: 8 192 elements, 327 680 DoFs**

# Residual-based refinement for the laminar delta wing (BTC3)

**Freestream conditions:** $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.

**Residual-based refinement**

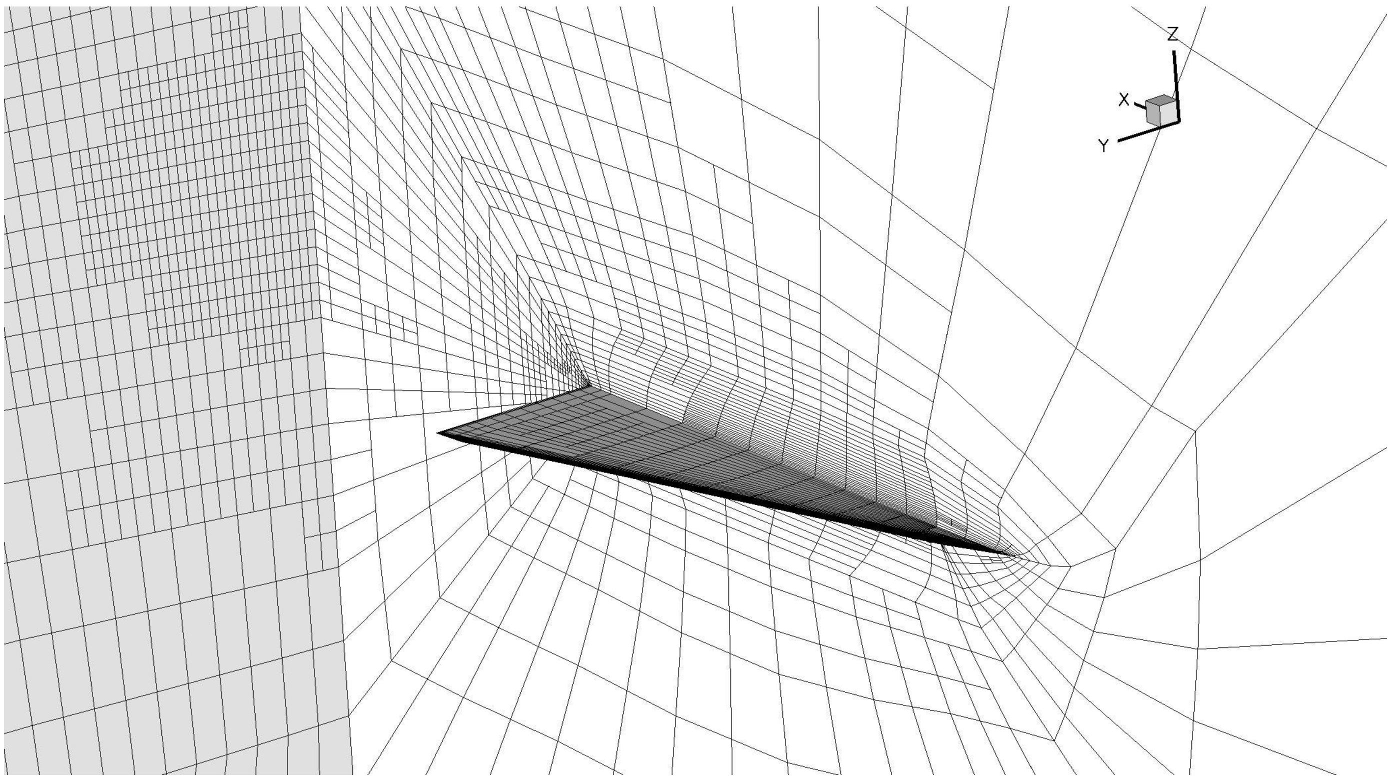



**2. refinement step: 21 352 elements, 854 080 DoFs**

# Residual-based refinement for the laminar delta wing (BTC3)

**Freestream conditions:** $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.

**Residual-based refinement**

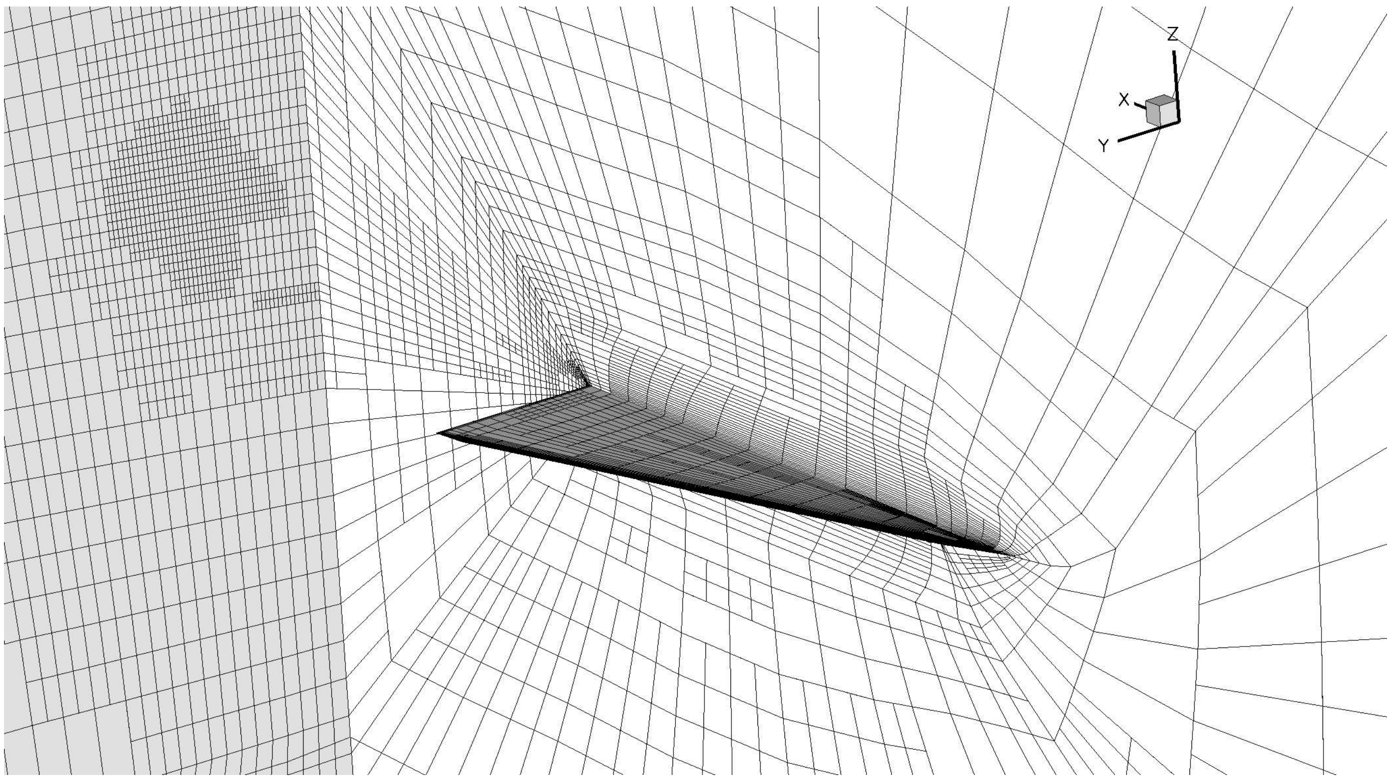



**3. refinement step: 55 673 elements, 2 226 920 DoFs**

# Residual-based refinement for the laminar delta wing (BTC3)

**Freestream conditions:** $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$.

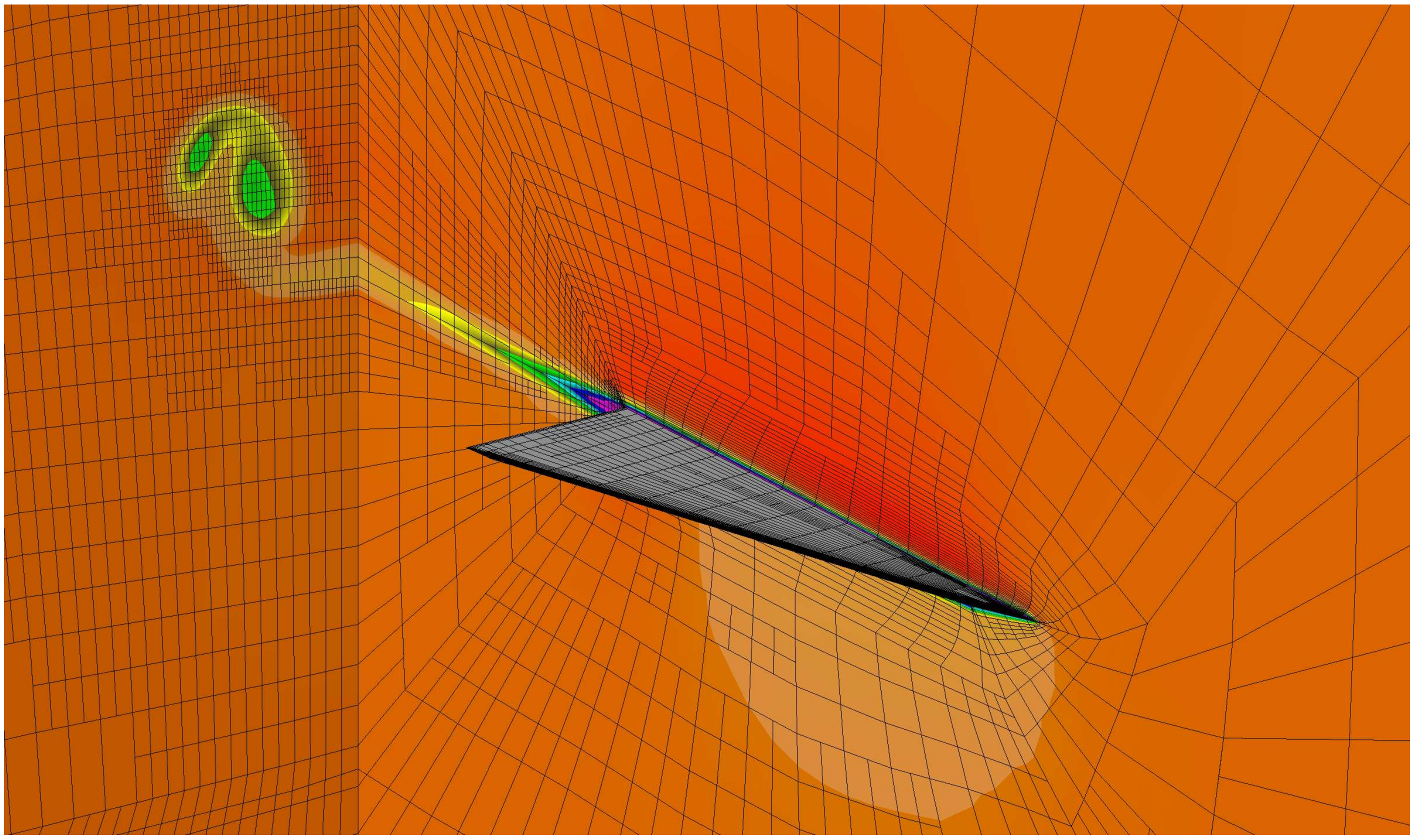
**Residual-based refinement**




**4. refinement step: 144 279 elements, 5 771 160 DoFs**

# Residual-based refinement for the laminar delta wing (BTC3)

**Freestream conditions:** $M = 0.3$, $\alpha = 12.5^\circ$, $Re = 4000$.



# Error estimation and goal-oriented (adjoint-based) refinement

**ADIGMA BTC3 test case: laminar flow around a delta wing.**

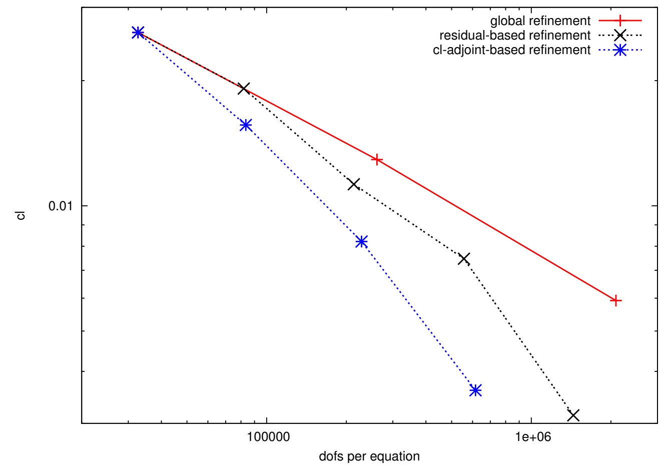
**Freestream conditions:** $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.

**Reference values:** $c_\mathrm{l}^{ref} = 0.3494$, $c_\mathrm{d}^{ref} = 0.1664$, $c_\mathrm{m}^{ref} = -0.0311$

$$J(\mathbf{u}) - J(\mathbf{u}_h) = \mathcal{R}(\mathbf{u}_h, \mathbf{z}) \approx \mathcal{R}(\mathbf{u}_h, \tilde{\mathbf{z}}_h) = \sum_{\kappa \in \mathcal{T}_h} \eta_\kappa,$$

| | | error in $c_l$ | | |
|---|---|---|---|---|
| cells | DoFs | exact | estimate | ratio |
| 3 264 | 130 560 | -2.611e-02 | -2.030e-02 | 0.78 |
| 8 346 | 333 840 | -1.564e-02 | -1.266e-02 | 0.81 |
| 22 843 | 913 720 | -8.209e-03 | -8.959e-03 | 1.09 |
| 61 567 | 2 462 680 | -3.603e-03 | -3.612e-03 | 1.00 |

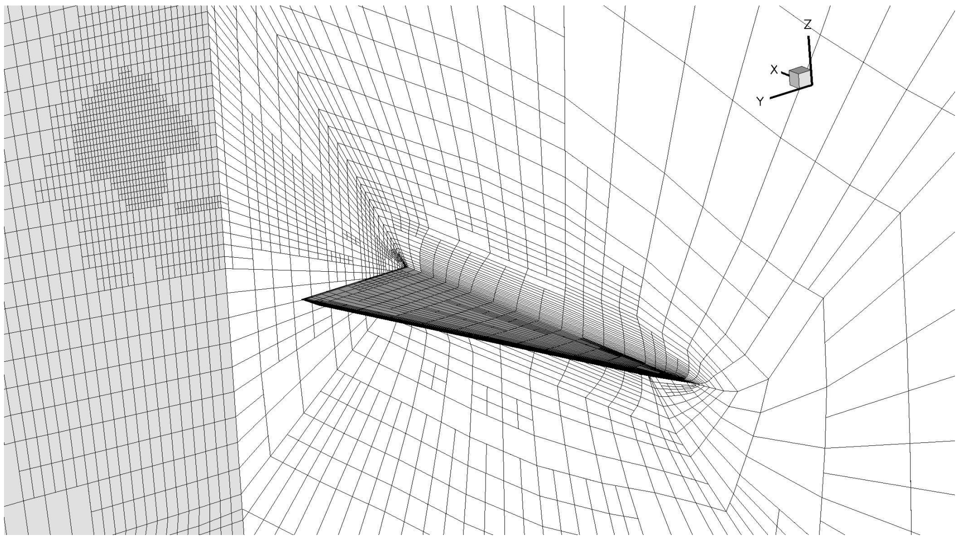
**Error in $c_l$**




**Similar for $c_d$ and $c_m$.**

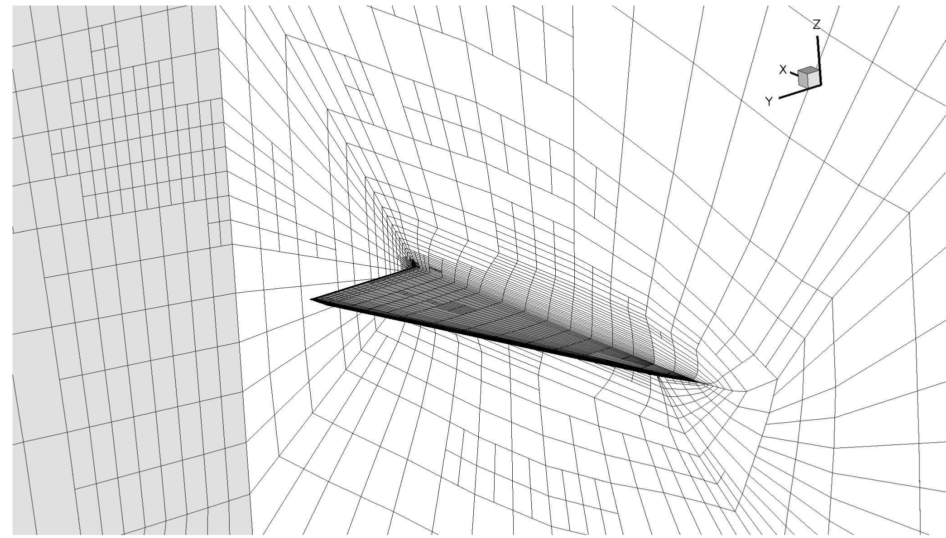
# Local refinement for the laminar delta wing (BTC3)

**Freestream conditions:** $M = 0.3$, $\alpha = 12.5°$, $Re = 4000$.




**residual-based refinement**

**5 771 160 DoFs, $c_l$: |error|=3.2e-03**




**adjoint-based refinement**

**2 462 680 DoFs, $c_l$: |error|=3.6e-03**

# Summary

## 2d and 3d laminar compressible flows

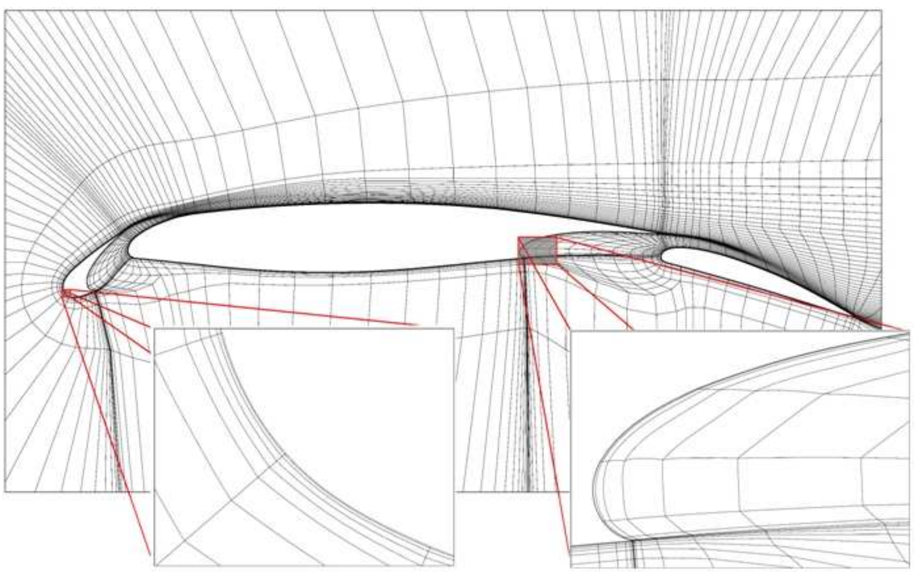
- Error estimation and goal-oriented (adjoint-based) refinement for single and for multiple aerodynamic force coefficients
- Residual-based mesh refinement for 3d laminar flows
- Error estimation and goal-oriented mesh refinement for 3d laminar flows

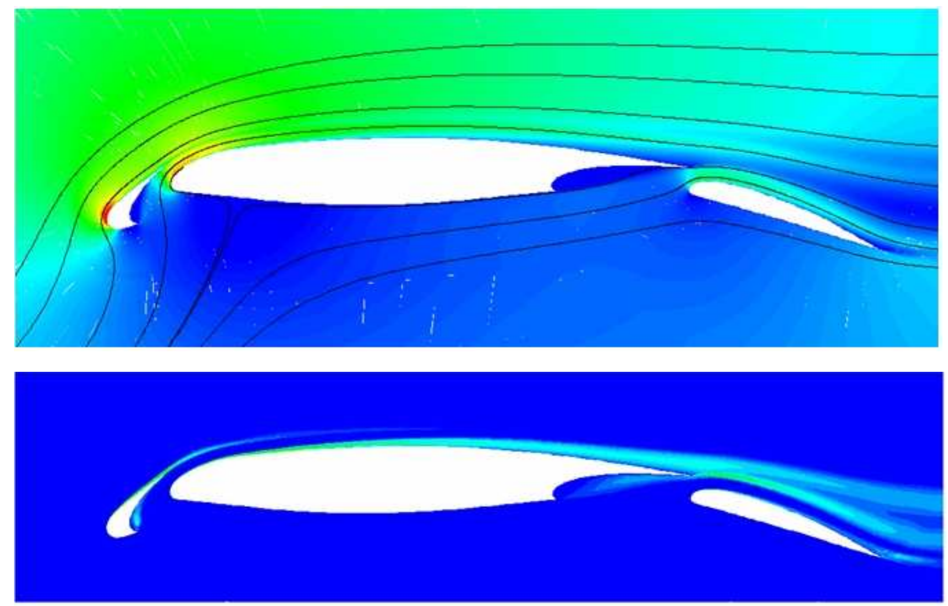
# Outlook

## Extension to turbulent flows

## Example: L1T2 three element airfoil (high lift configuration)

$M = 0.197, \alpha = 20.18^\circ, Re = 3.52 \cdot 10^6$, **4th order DG discretization**

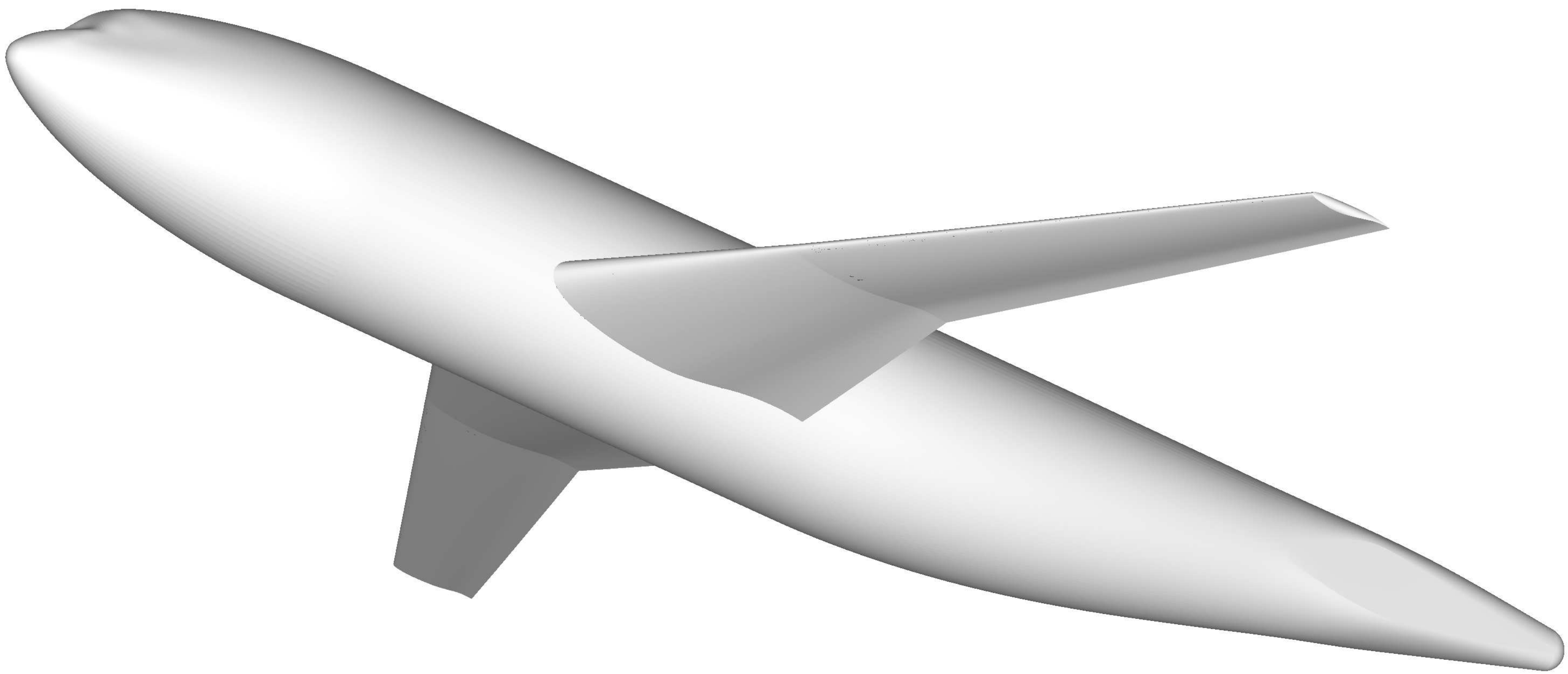




# Outlook

## Extension to complex test cases

**Example: DLR-F6 wing/body configuration without fairing**
**Geometry used in DPW II (the second drag prediction workshop)**



C²A²S²E Center for Computer Applications in AeroSpace Science and Engineering

ADIGMA

**Institute of Aerodynamics and Flow Technology, DLR, Braunschweig**

**The DG group:**

- **Ralf Hartmann**
- **Joachim Held**
- **Tobias Leicht**
- **Florian Prill**

# Thank you.