Computation of $\mathcal{L}_\infty$-norm of linear discrete-time periodic systems

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***Abstract*— We propose an efficient and numerically reliable procedure for the computation of $\mathcal{L}_\infty$-norm of discrete-time linear periodic systems. The new procedure is general being applicable to both standard as well as descriptor periodic systems with time-varying dimensions. The convergence rate of the iterative part is quadratic. The proposed procedure is based on structure preserving numerically stable algorithms for eigenvalue and zeros computation.**

I. Introduction

We consider periodic time-varying descriptor systems of the form

$$\begin{array}{rcl} E_k x(k+1) &=& A_k x(k) + B_k u(k) \\ y(k) &=& C_k x(k) + D_k u(k) \end{array} \qquad (1)$$

where the matrices $E_k \in \mathrm{R}^{\mu_k \times n_{k+1}}$, $A_k \in \mathrm{R}^{\mu_k \times n_k}$, $B_k \in \mathrm{R}^{\mu_k \times m_k}$, $C_k \in \mathrm{R}^{p_k \times n_k}$, $D_k \in \mathrm{R}^{p_k \times m_k}$ are periodic with period $N \geq 1$, and the dimensions fulfil the condition $\nu = \sum_{k=1}^{N} \mu_k = \sum_{k=1}^{N} n_k$. The periodic system (1) will be alternatively denoted by the periodic quintuple $(E_k, A_k, B_k, C_k, D_k)$.

To define the $\mathcal{L}_\infty$-norm of the periodic system (1), we define first the *transfer-function matrix* (TFM) corresponding to the associated *stacked lifted representation* of [1], which uses the input-state-output behavior of the system over time intervals of length N, rather then 1. For a given sampling time k, the corresponding M-dimensional input vector, P-dimensional output vector and ν-dimensional state vector are

$$\begin{array}{rcl} u_k^L(h) &=& [u^T(k+hN) \cdots u^T(k+hN+N-1)]^T, \\ y_k^L(h) &=& [y^T(k+hN) \cdots y^T(k+hN+N-1)]^T, \\ x_k^L(h) &=& [x^T(k+hN) \cdots x^T(k+hN+N-1)]^T. \end{array}$$

where $M = \sum_{k=1}^{N} m_k$ and $P = \sum_{k=1}^{N} p_k$. The corresponding *stacked* lifted system can be represented by a time-invariant descriptor system of the form

$$\begin{array}{rcl} L_k x_k^L(h+1) &=& F_k x_k^L(h) + G_k u_k^L(h) \\ y_k^L(h) &=& H_k x_k^L(h) + J_k u_k^L(h) \end{array} \qquad (2)$$

where

$$F_k - zL_k = \begin{bmatrix} A_k & -E_k & O & \cdots & O \\ O & A_{k+1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ O & O & \cdots & A_{k+N-2} & -E_{k+N-2} \\ -zE_{k+N-1} & O & \cdots & O & A_{k+N-1} \end{bmatrix} \qquad (3)$$

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$$G_k = \mathrm{diag}\,(B_k, \ldots, B_{k+N-1}),$$

$$H_k = \mathrm{diag}\,(C_k, \ldots, C_{k+N-1}),$$

$$J_k = \mathrm{diag}\,(D_k, \ldots, D_{k+N-1})$$

Assuming the square pencil (3) is regular (i.e., $\det(F_k - zL_k) \not\equiv 0$), the TFM of the lifted system is

$$W_k(z) = H_k(zL_k - F_k)^{-1}G_k + J_k \qquad (4)$$

and depends on the sampling time k.

For a given $\theta \in [\,0,\, 2\pi\,]$, the frequency-response matrix $W_k(e^{j\theta})$ is given by

$$W_k(e^{j\theta}) = H_k(e^{j\theta}L_k - F_k)^{-1}G_k + J_k$$

If T_s is the sampling period of the system, then θ can be expressed as $\theta = \omega T_s$.

Definition 1: Assume the TFM $W_k(z)$ has no poles on the unit circle. The $\mathcal{L}_\infty$-norm of the periodic system (1) is defined as the $\mathcal{L}_\infty$-norm of $W_k(z)$

$$\|W_k(z)\|_\infty := \max_{\theta \in [\,0,\, 2\pi\,]} \sigma_{max}\left(W_k(e^{j\theta})\right)$$

where $\sigma_{max}(\cdot)$ denotes the maximum singular value.

Since $W_{k+N}(z) = W_k(z)$ and the TFMs at two successive values of k are related by the relation [2]

$$W_{k+1}(z) = \begin{bmatrix} 0 & I_{P-p_k} \\ zI_{p_k} & 0 \end{bmatrix} W_k(z) \begin{bmatrix} 0 & z^{-1}I_{m_k} \\ I_{M-m_k} & 0 \end{bmatrix} \qquad (5)$$

the $\mathcal{L}_\infty$-norm is independent of the time index k. Therefore, to simplify the notation, we drop in what follows the dependence of index k of all matrices, by assuming $k = 1$.

The computation of the $\mathcal{L}_\infty$-norm can be done in several ways. A standard way is to evaluate $\|W(z)\|_\infty$ starting from its definition by computing the largest value of $\sigma_{max}\left(W(e^{j\theta})\right)$ for a sufficiently dense grid of points θ_i in $[\,0,\, 2\pi\,]$. To evaluate $W(e^{j\theta})$, the recently proposed efficient algorithms can be employed [3]. This method has the obvious limitation to possibly miss peak frequency points located between the grid points. A second approach for standard periodic systems with constant dimensions (i.e., $E_k = I$), exploits the fulfillment of condition $\|W(z)\|_\infty < \gamma$ provided a stabilizing solution of a certain periodic Riccati equation exists [4]. In fact, for norm computation it is sufficient to compute for a given value of γ the eigenvalues of an associated symplectic periodic pair and check the existence of eigenvalues lying on the unit circle. If such eigenvalues exists, the condition $\|W(z)\|_\infty < \gamma$ is not fulfilled and the condition can be

tested for a larger value of γ. This leads to a so-called γ-iteration based search (e.g., using bisection), which allows to compute the $\mathcal{L}_\infty$-norm of $W(z)$ to a desired accuracy. A similar approach has been proposed in [5] for the more general case of descriptor periodic systems with constant dimensions.

In this paper we propose a quadratically convergent algorithm to compute efficiently the $\mathcal{L}_\infty$-norm of a periodic system. The proposed method improves the method of [5] in several ways. First, the new method is applicable to arbitrary periodic descriptor systems with time-varying dimensions. Secondly, the new algorithm employs a better initial approximation of the norm by evaluating $W(z)$ in a set of selected points using the frequency-response computation algorithm of [6]. This has the main benefit of reducing significantly the number of subsequent iterations. In fact, in many cases the γ-iteration can be completely avoided by using this initialization. Finally, the algorithm has a true quadratic convergence, by employing the mid-point update strategy [7], [8].

II. $\mathcal{L}_\infty$-NORM COMPUTATION PROCEDURE

In this section we describe a conceptual procedure for the computation of the $\mathcal{L}_\infty$-norm of the periodic system (1). The procedure is presented using the associated lifted periodic descriptor system

$$W(z) = \left[\begin{array}{c|c} F - zL & G \\ \hline H & J \end{array}\right]$$

Details on how to implement this procedure efficiently by completely avoiding lifting are given in the subsequent sections.

Our approach is based on the following result [9]:

Theorem 1: If $\gamma > 0$ is not a singular value of J, then $\gamma^2 I - W(z)W^T(1/z)$ has a zero $e^{j\theta}$ with $\theta \in [\,0,\ 2\pi\,]$ if and only if γ is a singular value of $W(e^{j\theta})$.

Using this theorem, we can devise a procedure to compute the $\mathcal{L}_\infty$-norm which is based on computing repeatedly the finite poles and non-null zeros of

$$Z_\gamma(z) := \gamma^2 I - W(z)W^T(1/z) \tag{6}$$

The following norm computation procedure is similar to that proposed in [9]:

Algorithm. *$\mathcal{L}_\infty$-norm computation.*

Comment. Compute lower bound.

$\gamma_{lb} := \max\{\sigma_{max}\left(W(e^{j\theta_i})\right)\}$ for $\theta_i \in \{0, \pi, \theta_{p_j}, j = 1, \ldots, n_f\}$, where $e^{j\theta_{p_j}}$ is an approximation of the peaking frequency of a finite pole z_{p_j} of $W(z)$, $j = 1, \ldots, n_f$.

Comment. Perform γ-iteration with midpoint update.

repeat until **break**

  $\gamma = (1 + tol)\gamma_{lb}$
  compute $z_i = e^{j\theta_i}$, $i = 1, \ldots, k$ the zeros of $Z_\gamma(z)$ on the unit circle
  if $k = 0$, **then**
    break
  else
    $m_i = (\theta_i + \theta_{i+1})/2$, $i = 1, \ldots, k-1$
    $\gamma_{lb} := \max_i\{\sigma_{max}\left(W(e^{jm_i})\right)\}$
  end

end repeat

$\|W(z)\|_\infty = (\gamma_{lb} + \gamma)/2$

The midpoint updating employed to determine the new lower bound approximation γ_{lb} guarantees an *at least* quadratic convergence of this procedure [8]. The main computational ingredients for this procedure are the computation of finite poles, frequency-response and finite non-null zeros of certain descriptor systems. For all these computations, numerically reliable algorithms exists and robust numerical implementations are available in the DESCRIPTOR TOOLBOX FOR MATLAB [10]. However, this approach is a numerically naive way to compute the $\mathcal{L}_\infty$-norm of a periodic system because, by ignoring the problem structure, the computational complexity of about $O((2n + m + p)^3 N^3)$ is inadmissibly large (n, m, p are the maximal dimensions of state, input and output vectors). In what follows we discuss in some details structure exploiting algorithms which can be used to reduce the overall computational complexity of this procedure to an acceptable level [11].

III. ALGORITHMS FOR EIGENVALUES COMPUTATION

Eigenvalue computation is necessary to determine the initial guess for γ_{lb} by computing the finite poles of $W(z)$. This computation can be performed by computing the finite eigenvalues of the poles pencil $P_p(z) = F - zL$ in (3)

$$P_p(z) := \left[\begin{array}{ccccc} S_1 & -T_1 & O & \cdots & O \\ O & S_2 & -T_2 & \cdots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & & & S_{N-1} & -T_{N-1} \\ -zT_N & O & \cdots & O & S_N \end{array}\right] \tag{7}$$

where $S_i = A_i$ and $T_i = E_i$ for $i = 1, \ldots, N$. In what follows we present shortly two classes of structure exploiting algorithms which can be used to efficiently perform this computation. As it will be clear in section IV, the same algorithms can be also employed to compute

the finite non-null eigenvalues of the regular pencil $P_\gamma(z)$ (see (10)) used in the γ-iteration.

A. Fast algorithm

The idea of the fast algorithm is to reduce the pencil (7) by employing structure exploiting orthogonal reductions to a lower size one which has the same finite eigenvalues and for which standard pencil manipulation methods (e.g., staircase algorithm for finite-infinite spectral separation and QZ algorithm for eigenvalue computation) can be employed. The reduction method is a particular instance of a general reduction method employed for the computation of zeros of periodic descriptor systems [12] and is related to the *swapping and collapsing* approach proposed in [13] of quotient matrix products. The basic reduction can be performed as follows.

Consider the $(\mu_1+\mu_2)$-th order orthogonal matrix U_1 compressing the rows of the $(\mu_1+\mu_2)\times n_2$ full column rank matrix $\begin{bmatrix} -T_1 \\ S_2 \end{bmatrix}$ to $\begin{bmatrix} R_1 \\ O \end{bmatrix}$, where R_1 is a nonsingular matrix of order n_2. Applying U_1 to the first two block rows of $P_p(z)$ we obtain for the nonzero elements

$$U_1 \begin{bmatrix} S_1 & -T_1 & O \\ O & S_2 & -T_2 \end{bmatrix} = \begin{bmatrix} \widetilde{S}_1 & R_1 & -\widetilde{T}_1 \\ \widehat{S}_2 & O & -\widehat{T}_2 \end{bmatrix}$$

which defines the new matrices $\widehat{S}_2$ and $\widehat{T}_2$ with $\mu_1+\mu_2-n_2$ rows.

Then construct the $(\sum_{j=1}^{i+1}\mu_j - \sum_{j=2}^{i} n_j)$-th order orthogonal transformations U_i for $i=2,\ldots N-1$ such that

$$U_i \begin{bmatrix} \widehat{S}_i & -\widehat{T}_i & O \\ O & S_{i+1} & -T_{i+1} \end{bmatrix} = \begin{bmatrix} \widetilde{S}_i & R_i & -\widetilde{T}_i \\ \widehat{S}_{i+1} & O & -\widehat{T}_{i+1} \end{bmatrix}$$

where R_i are invertible matrices of order n_{i+1}. This recursively defines the new matrices $\widehat{S}_{i+1}$ and $\widehat{T}_{i+1}$ with $(\sum_{j=1}^{i+1}\mu_j - \sum_{j=2}^{i+1} n_j)$ rows.

Applying the transformations U_i successively to the i-th and $(i+1)$-th block rows of the transformed pencil $P_p(z)$, we obtain the reduced pencil

$$\overline{P}_p(z) = \left[\begin{array}{c|cccc} \widetilde{S}_1 & R_1 & -\widetilde{T}_1 & O & O \\ \widetilde{S}_2 & O & R_2 & \ddots & O \\ \vdots & \vdots & \vdots & \ddots & -\widetilde{T}_{N-2} \\ \widetilde{S}_{N-1} - z\widetilde{T}_{N-1} & O & O & \cdots & R_{N-1} \\ \hline \widehat{S}_N - z\widehat{T}_N & O & O & \cdots & O \end{array}\right] \quad (8)$$

which is orthogonally similar to $P_p(z)$. Since the matrices R_i are nonsingular, the regular subpencil $\widehat{S}_N - z\widehat{T}_N$ of order n_1 will contain all finite eigenvalues of the original pencil. To check the regularity of the extended pencil, the reciprocal condition numbers of the upper triangular matrices R_i can be cheaply estimated to detect possible rank losses.

To compute only the finite eigenvalues, orthogonal V and Z are determined (see for example [14]) such that

$$V(\widehat{S}_N - z\widehat{T}_N)Z = \begin{bmatrix} S^\infty - zT^\infty & S^{\infty,f} - zT^{\infty,f} \\ O & S^f - zT^f \end{bmatrix}$$

where $S^f - zT^f$ has only finite eigenvalues and $S^\infty - zT^\infty$ has only infinite eigenvalues. The finite eigenvalues are computed as the generalized eigenvalues of the pair (S^f, T^f) using the well-known QZ algorithm [15].

To estimate the computational effort of this approach, let assume constant dimensions n for the matrices $A_k, E_k \in \mathbb{R}^{n\times n}$. The reduction of the pencil $P_p(z)$ can be done by computing successively $N-1$ QR decompositions of $2n\times n$ matrices and applying the transformation to two subblocks of the same dimensions. The reduction step has thus a computational complexity of $O((N-1)n^3)$. Since the last step, the computation of finite-infinite spectral separation of the reduced pencil $\widehat{S}_N - z\widehat{T}_N$, has a complexity of $O(n^3)$, it follows that the overall computational complexity of the proposed approach corresponds to what is expected for a satisfactory algorithm for periodic systems [11]. This approach is substantially more efficient than the periodic QZ algorithm based approach (see next subsection) and this is why, the proposed algorithm belongs to the family of so-called *fast* algorithms [16].

Since the main reduction consists of successive QR-decompositions, it can be shown [17] that the matrices of the computed reduced pencil $\overline{P}_p(z) := \overline{F} - \lambda\overline{L}$ satisfy

$$\|UX - \overline{X}\|_2 \le \epsilon_M f(nN)\|X\|_2, \quad X = F,\ L$$

where U is the matrix of accumulated left orthogonal transformations, ϵ_M is the relative machine precision, and $f(nN)$ is a quantity of order of nN. The subsequent computational step is performed using the algorithm of [14] and is also based exclusively on orthogonal transformations. This second step is numerically stable as well. Overall, it is thus guaranteed that the computed finite eigenvalues are exact for a slightly perturbed pole pencil. It follows that the proposed algorithm to compute finite eigenvalues is *numerically backward stable*.

Since the structure of the perturbed pole pencil is not preserved in the reduction, we can not say however that the computed eigenvalues are exact for slightly perturbed original data (i.e., the algorithm is not *strongly* stable).

B. Strongly stable algorithm

The computation of finite eigenvalues can be performed alternatively using a structure preserving reduction of the pole pencil. This reduction can be performed in three steps. In the first step we perform the finite-infinite spectral separation of periodic pairs which can be computed using the recently developed algorithm to compute Kronecker-like forms of periodic pairs [18]. By applying this algorithm to the periodic pair (S_k, T_k), orthogonal matrices V_k^1 and Z_k^1 are determined such that

$$V_k^1 S_k Z_k^1 = \begin{bmatrix} S_k^\infty & S_k^{\infty,f} \\ O & S_k^f \end{bmatrix},$$

$$V_k^1 T_k Z_{k+1}^1 = \begin{bmatrix} T_k^\infty & T_k^{\infty,f} \\ O & T_k^f \end{bmatrix},$$

where T_k^f and S_k^∞ are nonsingular matrices. The pair (S_k^f, T_k^f) contains the finite characteristic multipliers, while the pair (S_k^∞, T_k^∞) contains the infinite characteristic multipliers. The finite characteristic multipliers at time k are the eigenvalues of the product $(T_{k+N-1}^f)^{-1} S_{k+N-1}^f \cdots (T_k^f)^{-1} S_k^f$. Note that the regularity of pencil $P_p(z)$ is implicitly checked by the existence of the above separation (i.e., the resulting Kronecker-like form contains no parts which correspond to a left or right Kronecker structure).

In the second step, orthogonal matrices V_k^2 and Z_k^2 are determined such that

$$V_k^2 S_k^f Z_k^2 = \begin{bmatrix} S_{k,11}^f & S_{k,12}^f \\ O & S_{k,22}^f \end{bmatrix},$$

$$V_k^2 T_k^f Z_{k+1}^2 = \begin{bmatrix} T_{k,11}^f & T_{k,11}^f \\ O & T_{k,22}^f \end{bmatrix},$$

where the periodic pair of square matrices $(S_{k,11}^f, T_{k,11}^f)$ has constant dimensions and is in a periodic generalized Hessenberg form [19], $T_{k,22}^f$ is upper triangular and $S_{k,22}^f$ is upper trapezoidal. The periodic pair $(S_{k,22}^f, T_{k,22}^f)$ has only null characteristic multipliers. This reduction can be performed by extending the generalized periodic Hessenberg reduction procedures of [20], [19] to the case of non-constant dimensions similarly as done in [21] for the periodic Hessenberg form.

Finally, the orthogonal matrices V_k^3 and Z_k^3 are computed such that

$$V_k^3 S_{k,11}^f Z_k^3 = S_k^s, \quad V_k^3 T_{k,11}^f Z_{k+1}^3 = T_k^s,$$

where the pair (S_k^s, T_k^s) is in a generalized periodic real Schur form. For this step, the algorithms of [20], [19] to compute the periodic QZ decomposition can be used.

The performed orthogonal transformations are not needed to be accumulated when computing only the eigenvalues. Each computational step of the above algorithm is strongly numerically stable and has a computational complexity of $O(n^3 N)$, where n is the maximum of problem dimensions for n_k. The increased numerical reliability is however achieved with a significantly larger computational cost than for the fast algorithm of the previous subsection. In the case of a standard periodic system (i.e., $E_k = I_{n_{k+1}}$), obvious simplifications arise. In this case, the periodic Schur algorithms [20], [19] in conjunction with the reduction to the extended periodic Hessenberg form [21] can be employed to compute the eigenvalues of the matrix product $A_N \cdots A_2 A_1$ without actually forming this product.

IV. Computation of zeros

To compute the zeros of $Z_\gamma(z)$ in (6), ideally we would like to use structure exploiting algorithms to compute zeros of periodic systems like those proposed in [12], [22]. Therefore, all what we have to do is to build a periodic realization of $Z_\gamma(z)$. We can easily see that the descriptor system realization of $W^T(1/z)$

$$W^T(1/z) = \left[\begin{array}{cc|c} L^T - zF^T & O & -H^T \\ zH^T & -I & J^T \\ \hline O & I & O \end{array}\right] \tag{9}$$

does not correspond in general to a lifted descriptor representation. However, the next result shows that we can turn this realization into one which corresponds to a periodic system realization.

Proposition 1: Assume the rational matrix $G(z)$ has an N-periodic descriptor realization $(E_k, A_k, B_k, C_k, D_k)$. Then, $G^T(1/z)$ has a periodic realization given by $(\widetilde{E}_k, \widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k, \widetilde{D}_k)$, where

$$\widetilde{E}_k = \begin{bmatrix} A_k^T & O_{n_k \times m_{k+1}} \\ -B_k^T & O_{m_k \times m_{k+1}} \end{bmatrix}, \quad \widetilde{A}_k = \begin{bmatrix} E_{k-1}^T & O_{n_k \times m_k} \\ O_{m_k \times \mu_{k-1}} & -I_{m_k} \end{bmatrix},$$

$$\widetilde{B}_k = \begin{bmatrix} -C_k^T \\ D_k^T \end{bmatrix}, \ \widetilde{C}_k = \begin{bmatrix} O_{m_k \times \mu_{k-1}} I_{m_k} \end{bmatrix}, \ \widetilde{D}_k = O_{m_k \times p_k}$$

The proof of this result is straightforward and relies on the observation that the realization (9) is non-minimal, containing artificially induced unobservable eigenvalues and zeros in the origin which cancel each others.

Let $(\widetilde{F} - z\widetilde{L}, \widetilde{G}, \widetilde{H}, 0)$ be the lifted descriptor system corresponding to the conjugated TFM $W^T(1/z)$. It follows that the zeros of $Z_\gamma(z)$ are the generalized eigenvalues of the system pencil

$$P_\gamma(z) = \begin{bmatrix} F - zL & G\widetilde{H} & O \\ O & \widetilde{F} - z\widetilde{L} & -\widetilde{G} \\ H & J\widetilde{H} & \gamma^2 I \end{bmatrix} \tag{10}$$

By reordering the blocks of $P_\gamma(z)$ we can put this pencil in the cyclic form (7), with the N-periodic matrix pair (S_k, T_k) defined as

$$S_k = \begin{bmatrix} A_k & O & B_k & O \\ O & E_{k-1}^T & O & C_k^T \\ O & O & I_{m_k} & D_k^T \\ C_k & O & D_k & \gamma^2 I_{p_k} \end{bmatrix}$$

$$T_k = \begin{bmatrix} E_k & O & O & O \\ O & A_k^T & O & O \\ O & B_k^T & O & O \\ O & O & O & O \end{bmatrix}$$

Thus, to compute the zeros of $Z_\gamma(z)$, the eigenvalue algorithms of previous section can be applied to the pair (S_k, T_k).

Computational effort saving results if we exploit the structure of this pair by isolating a part of non-dynamic infinite eigenvalues. This can be done using invertible matrices Q_k, $k = 1, \ldots, N$ which compress the rows in the last two columns of S_k to full rank matrices

$$Q_k \begin{bmatrix} B_k & O \\ O & C_k^T \\ I_{m_k} & D_k^T \\ D_k & \gamma^2 I_{p_k} \end{bmatrix} = \begin{bmatrix} O \\ R_k \end{bmatrix}. \tag{11}$$

By applying this transformation to S_k and T_k, we obtain

$$Q_k S_k = \begin{bmatrix} \overline{S}_k & O \\ * & R_k \end{bmatrix}, \quad Q_k T_k = \begin{bmatrix} \overline{T}_k & O \\ * & O \end{bmatrix}$$

where the reduced pair $(\overline{S}_k, \overline{T}_k)$ has the same finite eigenvalues as the original pair (S_k, T_k).

For example, since

$$R_k := \begin{bmatrix} I_{m_k} & D_k^T \\ D_k & \gamma^2 I_{p_k} \end{bmatrix}$$

are invertible provided γ is not a singular value of D_k, we can choose

$$Q_k = \begin{bmatrix} \begin{bmatrix} I & O \\ O & I \end{bmatrix} & -\begin{bmatrix} B_k & O \\ O & C_k^T \end{bmatrix} R_k^{-1} \\ \begin{bmatrix} O & O \\ O & O \end{bmatrix} & \begin{bmatrix} I & O \\ O & I \end{bmatrix} \end{bmatrix}$$

We obtain

$$\overline{S}_k = \begin{bmatrix} A_k & O \\ O & E_{k-1}^T \end{bmatrix} - \begin{bmatrix} B_k & O \\ O & C_k^T \end{bmatrix} R_k^{-1} \begin{bmatrix} O & O \\ C_k & O \end{bmatrix}$$

$$\overline{T}_k = \begin{bmatrix} E_k & O \\ O & A_k^T \end{bmatrix} - \begin{bmatrix} B_k & O \\ O & C_k^T \end{bmatrix} R_k^{-1} \begin{bmatrix} O & B_k^T \\ O & O \end{bmatrix}$$

These matrices are the same as those deduced in [5], where a minor index shift error is present (E_k stays instead of E_{k-1} in the expression of $\overline{S}_k$). Since Q_k are nonorthogonal, this reduction can induce potential errors when γ is near to a singular value of D_k. For a reliable numerical implementation, it is however necessary to perform the reduction in (11) using orthogonal Q_k obtained from QR decompositions. The algorithms of previous section can be applied to the reduced pairs $(\overline{S}_k, \overline{T}_k)$. The resulting overall computational complexity is $O((2n+m+p)^3 N)$, and thus acceptable according to [11].

V. Computation of frequency responses

To evaluate $W(\lambda_i)$ for $\lambda_i = e^{j\theta_i}$, the main computation is to solve the linear system $\widetilde{P}RX = \widetilde{P}G$, where $R = \lambda_i L - F$, and $\widetilde{P}$ is a permutation matrix which brings the last block row of R and G in the first block row. This is a potentially large order structured linear system known also (see [23]) as *bordered almost block diagonal* (BABD) with the coefficient matrix of the form

$$\widetilde{P}R = \begin{bmatrix} R_{11} & & & R_{1,N} \\ R_{21} & R_{22} & & \\ & \ddots & \ddots & \\ & & R_{N,N-1} & R_{N,N} \end{bmatrix}$$

where $R_{11} = \lambda_i E_1$, $R_{kk} = E_k$ for $k = 2, \ldots, N$, $R_{i+1,i} = -\widehat{A}_i$ for $i = 1, \ldots, N-1$, $R_{1,N} = -A_N$,

$$\widetilde{P}G = \begin{bmatrix} & & & G_{1N} \\ G_{21} & & & \\ & \ddots & & \\ & & G_{N,N-1} & \end{bmatrix}$$

where $G_{i+1,i} = B_i$, for $i = 1, \ldots, N-1$ and $G_{1,N} = B_N$. The solution X, of the same size as G, is obtained in a $N \times N$ block partitioned form with the $\mu_i \times m_i$ matrix X_{ij} as its generic block element for $i, j = 1, \ldots, N$. For the sake of simplicity, we redefine $\widetilde{P}R$ and $\widetilde{P}G$ as R and G, respectively.

To solve the BABD system of equations $RX = G$ we describe shortly the structured *Gaussian elimination with partial pivoting* (GEPP). Recall that the standard GEPP method to solve the linear equation $RX = G$ has two main steps [17]. First, the LU factorization of R is computed by using partial row pivoting, to obtain $PR = LU$, where P is a row permutation matrix, L is a unit lower triangular matrix and U is an upper triangular matrix. Then, the solution is determined by using forward substitution to compute $Y = L^{-1}(PG)$ and back substitution to compute $X = U^{-1}Y$.

For the particular structure of R above, the resulting U has an upper triangular block structured form

$$U = \begin{bmatrix} U_{11} & U_{12} & & & U_{1N} \\ & U_{22} & U_{23} & & U_{2N} \\ & & \ddots & \ddots & \\ & & & \ddots & U_{N-1,N} \\ & & & & U_{N,N} \end{bmatrix} \tag{12}$$

with nonzero blocks only on the main block-diagonal, first block-supradiagonal and last block-column. Regarding L, nothing can be said in general about its bandwidth, but still, each column of the lower triangular part contains at most $2n$ nonzero elements.

As a consequence of the cyclic structure of G above, both $Y = L^{-1}(PG)$ and $Y = Q^T G$ have a special almost lower triangular structure with a nonzero last block column

$$Y = \begin{bmatrix} Y_{11} & & & Y_{1,N} \\ Y_{21} & Y_{22} & & Y_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{N,1} & Y_{N,2} & \cdots & Y_{N,N} \end{bmatrix} \tag{13}$$

Still, the resulting $X = U^{-1}Y$ is generally full, although to compute it the block structure of U can be efficiently exploited when computing it using block back-substitution. For the efficient evaluation of the matrix expression $HX + J$, the block-diagonal structure of the matrices H and J must be exploited.

Algorithmic details on the evaluation of $W(\lambda_i)$ are given in a separate paper [3]. The overall algorithm exploiting completely the problem structure requires

$$N_{op} = \left(\frac{23}{3}n^3 - \frac{3}{2}n^2 m\right) N + \left(\frac{9}{2}n^2 m + 2pmn\right) N^2$$

floating-point operations (i.e., additions/substractions and multiplication/divisions). Algorithmic enhancements for the case of standard periodic systems are discussed in [3]. For example, using a preliminary reduction of the periodic state matrix A_k to a periodic Hessenberg form, the number of operations can be reduced to

$$N_{op} = 2N^2 pmn + Nmn^2$$

while the preliminary reduction requires

$$N_{op,0} = \left(\frac{n^2 m}{2} + nmp\right) N^2 + \left(\frac{11}{3} n^3 + 3(p + \frac{m}{2}) n^2 - npm\right) N$$

operations. Note however, that this reduction must be performed only once.

VI. Conclusions

We proposed a numerically reliable algorithm with quadratic convergence to compute the $\mathcal{L}_\infty$-norm of linear discrete-time periodic system. The algorithm is completely general, being applicable to systems with time varying dimensions and even to periodic descriptor systems. An implementation of the proposed method is available in the PERIODIC TOOLBOX FOR MATLAB [24]. This implementation uses fast algorithms for eigenvalue and zeros computations.

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