OPTIMAL DISCRETE-TIME DESIGN OF MAGNETIC ATTITUDE CONTROL LAWS

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ABSTRACT

The problem of designing discrete-time attitude controllers for magnetically actuated spacecraft is considered. Several methods are discussed and a novel approach to the tuning of various classes of "projection based" controllers is proposed relying on periodic optimal output feedback control techniques. The main advantages of the proposed methods are discussed and illustrated in a simulation study.

1. INTRODUCTION

Electromagnetic actuators are a particularly effective and reliable technology for the attitude control of small satellites. As is well known, such actuators operate on the basis of the interaction between the magnetic field generated by a set of three orthogonal, current-driven coils and the magnetic field of the Earth and therefore provide a very simple solution to the problem of generating torques on board of a satellite. More precisely, magnetic torquers can be used either as main actuators for attitude control in momentum biased or gravity gradient attitude control architectures or as secondary actuators for momentum management tasks in zero momentum reaction wheel based configurations.

The main difficulty in the design of magnetic attitude control laws is related to the fact that magnetic torques are instantaneously constrained to lie in the plane orthogonal to the local magnetic field vector. Note that controllability of the attitude dynamics is ensured for a wide range of orbit altitudes and inclinations in spite of this constraint, thanks to the variability of the geomagnetic field, however, the control designer has to resort to time-varying control laws to deal with such effects.

In recent years, a considerable effort has been devoted to the analysis of this control problem (see, *e.g.*, (Arduini and Baiocco, 1997; Wisniewski and Blanke, 1999)); in particular, as the variability of the geomagnetic field is *almost* time-periodic, most of the recent work on the linear attitude control problem has focused on the use of optimal and robust periodic control theory for the design of state and output feedback regulators (Pittelkau, 1993; Varga and Pieters, 1998; Wisniewski and Markley, 1999; Lovera *et al.*, 2002; Psiaki, 2001; Lovera, 2001). See also (Silani and Lovera, 2005) for a recent survey on this subject. However, in spite of the extensive activity, the development of a design technique leading to a simple, easily implementable, yet efficient controller remains an open problem.

The aim of this paper (see also the preliminary results reported in (Lovera and Varga, 2005)) is to propose and compare a number of different approaches to the design of digital attitude controllers for spacecraft equipped with magnetic actuators, with specific emphasis on practical aspects associated with their on-board implementation. In particular, the results obtained using periodic optimal state feedback control are compared with the ones provided by a novel approach to the tuning of a class of fixed

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structure controllers for magnetic attitude control known as the "projection based" controllers.

It is important to note that the magnetic attitude control design problems associated with periodic optimisation techniques pose a significant challenge from the numerical point of view (possibly unstable open loop dynamics and very large period) and could be only solved by using reliable numerical methods as those implemented in the Periodic Systems Toolbox for Matlab (Varga, 2005*b*).

The paper is organised as follows. Section 2 provides a descrition of the spacecraft considered in the study as well as the derivation of a linearised model for its attitude dynamics. The considered control design techniques are subsequently described in Section 3, while the results obtained in the simulation of the designed control laws are presented and discussed in Section 4.

2. LINEARISED ATTITUDE DYNAMICS

In this study we will consider a spacecraft with inertia matrix $I = \mathrm{diag}\begin{bmatrix} I_{xx} & I_{yy} & I_{zz}\end{bmatrix}$, equipped with a single momentum/reaction wheel aligned with the body z axis, with moment of inertia J and angular velocity Ω relative to the body frame. For this spacecraft configuration the aim of the attitude control scheme is to maintain the spacecraft (body axes) aligned with the orbital axes, while exploiting the gyroscopic effect due to the momentum wheel. In the following we will derive linearised dynamic models for the formulation of this control problem.

Define the state vector $x_c = [\delta q_R^T \ \delta\omega^T]^T$ formed with small displacements of the vector part q_R of the attitude quaternion with respect to the orbital axes from the nominal values $\bar{q}_R = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^T$ and small deviations of the body rates from the nominal values $\overline{\omega}_x = \overline{\omega}_y = 0, \overline{\omega}_z = -\Omega_0$. Then the attitude dynamics can be linearized and the local linear dynamics for the attitude can be defined as

$$\dot{x}_c(t) = A_c x_c(t) + B_{cT}\left[T_{c,mag}(t) + T_{c,dist}(t)\right] \quad (1)$$

or

$$\dot{x}_c(t) = A_c x_c(t) + B_{cm}(t) m_c(t) + B_{cT} T_{c,dist}(t) \quad (2)$$

where

$$A_c = \begin{bmatrix} 0 & -\Omega_0 & 0 & 0.5 & 0 & 0 \\ \Omega_0 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & W_x & 0 \\ 0 & -6k_y\Omega_0^2 & 0 & W_y & 0 & 0 \\ 0 & 0 & +6k_z\Omega_0^2 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{cT} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_{xx}^{-1} & 0 & 0 \\ 0 & I_{yy}^{-1} & 0 \\ 0 & 0 & I_{zz}^{-1} \end{bmatrix}, \quad B_{cm}(t) = B_{cT} S(b(t)),$$

$$S(\omega) = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix},$$

and $k_x = \frac{I_{yy}-I_{zz}}{I_{xx}}, k_y = \frac{I_{zz}-I_{xx}}{I_{yy}}, k_z = \frac{I_{xx}-I_{yy}}{I_{zz}}$, $W_x = -k_x\Omega_0 - k_{wx}\overline{\Omega}$, $W_y = -k_y\Omega_0 + k_{wy}\overline{\Omega}$, $k_{wx} = \frac{J}{I_{xx}}, k_{wy} = \frac{J}{I_{yy}}$. Here, $\overline{\Omega}$ is the nominal wheel speed.

Note that two different control matrices B_{cT} and $B_{cm}(t)$ have been defined, in order to handle problem formulations in which either magnetic torques ($T_{c,mag}$) or magnetic dipoles (m_c) are used as control variables, respectively. Therefore, while A_c is constant, the control matrix $B_{cm}(t)$ corresponding to the control input m_c turns out to be time-varying (and approximately time-periodic with period $2\pi/\Omega_0$) because of the dependence on the geomagnetic field vector $b(t)$.

Finally, since we are concerned with a discrete-time design problem, suitable discrete-time equivalents of (1) and (2) have been derived, in the forms

$$x(k+1) = Ax(k) + B_T\left[T_{mag}(k) + T_{dist}(k)\right] \quad (3)$$

$$x(k+1) = Ax(k) + B_m(k)m(k) + B_T T_{dist}(k) \quad (4)$$

respectively, where for a sampling-time of $\Delta = 2\pi/(N\Omega_0)$ (N is the discrete-time period) we have

$$A := \exp(A_c\Delta) \quad (5)$$

$$B_T := \int_0^\Delta e^{A_c(\Delta-\tau)} B_{cT}\, d\tau \quad (6)$$

$$B_m(k) := \int_{k\Delta}^{(k+1)\Delta} e^{A_c[(k+1)\Delta-\tau]} B_{cm}(\tau)\, d\tau \quad (7)$$

$$x(k) := x_c(k\Delta) \quad (8)$$

$$T_{mag}(k) := T_{c,mag}(k\Delta), \quad T_{dist}(k) := T_{c,dist}(k\Delta) \quad (9)$$

and $m(k) := m_c(k\Delta)$.

3. CONTROLLER DESIGN

3.1 *Periodic optimal state feedback controller*

Consider the system (4) and let $u(k) = m(k)$. Minimizing the linear-quadratic (LQ) criterion

$$J = \sum_{k=0}^{\infty}\left[x(k)^T Q x(k) + u(k)^T R u(k)\right] \quad (10)$$

where $Q \geq 0$, $R > 0$ are symmetric matrices, is an attractive method to determine stabilizing periodic state feedback controllers of the form

$$u(k) = F(k)x(k) \quad (11)$$

The optimal N-periodic state-feedback matrix $F(k)$ minimizing the performance index (10) is given by

$$F(k) = -(R+B_m^T(k)X(k+1)B_m(k))^{-1}B_m^T(k)X(k+1)A$$

where the N-periodic symmetric positive semi-definite matrix $X(k)$ satisfies the reverse discrete-time periodic Riccati equation

$$X(k) = Q + A^T X(k+1)A + \\ -A^T X(k+1)B_m(k)(R + B_m^T(k)X(k+1)B_m(k))^{-1} \\ \cdot B_m^T(k)X(k+1)A$$

This periodic Riccati equation can be solved using the algorithm proposed in (Varga, 2005*a*) implemented in the Periodic Systems Toolbox (Varga, 2005*b*).

The optimal periodic LQ approach has the obvious advantage of providing a controller with a very good level of performance. This is the reason why optimal periodic control has been extensively studied as a viable approach to this problem, in a number of different settings and formulations: continuous-time in (Pittelkau, 1993; Wisniewski and Markley, 1999; Lovera *et al.*, 2002; Lovera, 2001) and discrete-time in (Wisniewski and Stoustrup, 2002). The issues associated with the implementation of optimal periodic controllers, however, make their actual application in real satellite missions not very likely: the storage requirements for a fully time-periodic gain are indeed a critical problem. While these issues motivate the interest in alternative approaches to this design problem, the performance level provided by the optimal periodic LQ controller can be taken as a reference for all other approaches.

3.2 *Fixed structure projection based controllers*

A very common approach to the design of attitude controllers for magnetically actuated satellites of the form (2) is to consider discrete-time control laws of the kind

$$u(k) = m(k) = -S(b(k\Delta))^T T_{id}(k), \qquad (12)$$

where $T_{id}(k)$ is an "ideal" control torque to be determined on the basis of a suitable static or dynamic feedback of state or output variables, according to the specific attitude control architecture of the considered spacecraft. Some examples of possible controller structures corresponding to equation (12) are the following:

- Static state feedback controller:

$$T_{id}(k) = K_x x(k), \qquad (13)$$

- Static output feedback controller:

$$T_{id}(k) = K_y y(k), \qquad (14)$$

  where y is given by, *e.g.*, a subset of the state variables or a given set of vector measurements.
- Dynamic output feedback controller:

$$z(k+1) = F_z z(k) + G_z y(k) \qquad (15)$$
$$T_{id}(k) = K_z z(k). \qquad (16)$$

Note that the advantage of the considered controller structures is that only constant parameters have to be designed, while the time-dependence of the control law is carried by the (measurable) value of the geomagnetic field b entering equation (12). However, to the best knowledge of the authors, no design approaches to the selection of the parameters in the proposed control laws (12) are available.

In this paper, we propose to face this design problem using the approach to the solution of optimal periodic output feedback problems first presented in (Varga and Pieters, 1998). This approach relies on a gradient-based optimization approach to determine time-periodic output feedback controllers by minimising the quadratic cost function (10).

The application of the results presented in (Varga and Pieters, 1998) to this problem requires a way of designing an initial stabilising gain, in order to reduce the numerical difficulties associated with open loop unstable dynamics, and to facilitate the convergence of the iterative optimization procedure. To this purpose, the initial gain of the controller has been selected according to the guidelines provided by (Lovera and Astolfi, 2004, Proposition 1) for the globally stabilising tuning of state feedback magnetic attitude controller of the "projection" type (*i.e.*, equation (12)).

>From a numerical point of view, the optimal tuning of the proposed control law (12) can be determined using a suitable function available in the Periodic Systems Toolbox (Varga, 2005*b*). This function is based on a gradient-based function minimization technique for problems with simple bounds (limited memory BFGS). To achieve the highest efficiency, the function and gradient evaluations have been implemented as a Fortran 95 *mex*-function based on the formulas derived in (Varga and Pieters, 1998).

4. SIMULATION RESULTS

In this Section, the performance of the considered control laws will be discussed in a detailed simulation study, and the results will be compared to those provided by the reference optimal periodic state feedback (PSF) LQ control strategy. For all the control laws, two values for the number of sampling points N over one orbit have been considered, namely $N = 100$ and $N = 300$, corresponding respectively to a sample interval Δ of about 56.1 and 18.7 seconds.

The considered spacecraft is of the type described in Section 2 and operates in a near polar orbit (87^o inclination) with an altitude of 450 Km and a corresponding orbital period of 5614.8 seconds. The numerical values of the parameters used in the mathematical model are:

- Satellite inertia$(kgm^2): I = diag$ [35 17 25];
- Momentum wheel inertia $(kgm^2): J = 0.01$;
- Orbital angular rate $(rad/s): \Omega_0 = 0.00111904$;
- Nominal wheel speed $(rad/s): \bar{\Omega} = 200$;

- Nominal (periodic) magnetic field components (Tesla), used for design purposes only:

$$b(t) = 10^{-6} \begin{bmatrix} 7\,cos(\Omega_0 t) + 48\,sin(\Omega_0 t) \\ 23\,cos(\Omega_0 t) - 2\,sin(\Omega_0 t) \\ 5 \end{bmatrix}$$

Magnetic coils with a saturation limit of $\pm 20\,Am^2$ have been considered.

The simulations have been carried out using an object-oriented environment for satellite dynamics developed using the Modelica language (see, *e.g.*, (Lovera, 2003; Pulecchi and Lovera, n.d.)). In particular, a nonlinear model for the spacecraft has been considered and the effect of gravity gradient torques (including J_2 effects) and of magnetic disturbance torques (such as due to a residual magnetic dipole of $1Am^2$ along each spacecraft body axis) have been taken into account.

Different control strategies, using the fixed structure projection approach, were adopted, namely:

- Static state feedback control (*SSF*).
- Static output feedback control with output consisting of measured angular rates, and pitch and roll angles (quaternion components q_3 and q_4) only, (*SOF*1).
- Static output feedback control, with output consisting of measured quaternion only (*SOF*2).
- Dynamic output feedback control, where the output is assumed to consist of the measured quaternion and a re-constructed angular rate vector. Two design approaches have been analyzed: a kinematic reconstruction of the angular rate vector alone (*DOF*1) and an on line estimation of the whole system state via Kalman filtering (*DOF*2). In the former case, the state has been augmented with an additional variable $z \in R^3$ such that the system (2) is now in the form

$$x_a = [\delta q_R^T \quad \delta\omega_R^T \quad z^T]^T$$
$$\dot{x_a} = \begin{bmatrix} A_C & & O_{6,3} \\ \frac{1}{T_{pd}} I_3 & O_3 & -\frac{1}{T_{pd}} I_3 \end{bmatrix} x_a + \begin{bmatrix} B_C \\ O_{3x6} \end{bmatrix} u \tag{17}$$

and the measures available are the spacecraft attitude and pseudo angular rates, computed as

$$\delta\hat{\omega}_R = 2\,(\frac{1}{T_{pd}}(\delta q_R - z) - M\delta q_R) \tag{18}$$

Here, T_{pd} is the time constant of the pseudo-derivative filter, set to a tenth of the sampling interval Δ. In the latter case, the whole system state (attitude and angular rates) has been estimated on line via a Kalman filter processing the satellite's attitude measured data. The Altair-HB Star Tracker performances have been taken as a reference for simulation analysis purpose.

All the proposed control laws have been designed using the Periodic Systems Toolbox for MATLAB. The weighting matrices in the quadratic cost function (10) have been chosen as $Q = 0.01\,I_n$ and $R = 100\,I_3$, where n is the state dimension.

The results obtained for the proposed controller designs can be summarized as follows:

- Closed loop stability: as can be inferred from Table 1, all the designed controllers lead to asymptotically stable closed loop dynamics.
- Optimality: the results obtained using periodic optimal control and the fixed structure controllers can be compared directly in terms of the achieved optimal values of the cost function (10). The performance loss associated with the adoption of the fixed structure controller instead of the optimal periodic one turns out to be acceptable for the constant state feedback, *SOF*1 and *DOF*2 design cases, both for $N = 100$ and $N = 300$. On the other hand, the *SOF*2 and *DOF*1 designs do not seem to be able to cope with the LQ-optimal reference design level of performance in terms of cost function, while they provide a satisfactory stability degree.

  The *SOF*2 controller provides surprisingly good performance, given the lack of information on the spacecraft angular rates. This result can be interpreted as follows. While in the continuous-time domain such a control strategy would not succeed in damping the spacecraft angular motion, the discrete-time implementation introduces an artificial damping in the system, as thoroughly discussed in (Kabamba, 1987). The achieved stability degree is strictly dependent upon the adopted sampling time Δ. For this reason, a simulation campaign has been carried on, with the purpose of optimally tuning the control law. As can be seen from Figure 1, the discrete-time period $N = 100$ leads to very good results, however the dependance of the controller performance from the chosen discrete-time period N is highly nonlinear, and necessitate of time consuming ad hoc tuning. This non linear dependency is easily shown in a simple case study, first presented in (Kabamba, 1987), reported for ease of discussion in the Appendix. In addition, as discussed in (Feuer and Goodwin, 1994), this approach has the drawback that, for small sampling times Δ, additional high frequency components are generated, centered on multiples of the sampling rate $1/\Delta$. When the output is sampled, these high-frequency components are folded back into the base-band frequency range, resulting in a modified sampled frequency response. As a consequence of having non negligible output power at high frequencies, the input power at those high frequency is even greater, and decreasing with the sampling rate. Anyway, in our case study the sampling frequency is very small (the magnitude of Δ being not smaller then 18.7 s), and so result the input high frequency harmonics. Therefore, the

*SOF*2 provides a simple and efficient way to guarantee closed loop stability and acceptable performance in the case of slow sampling rates, without requiring any angular rate feedback.

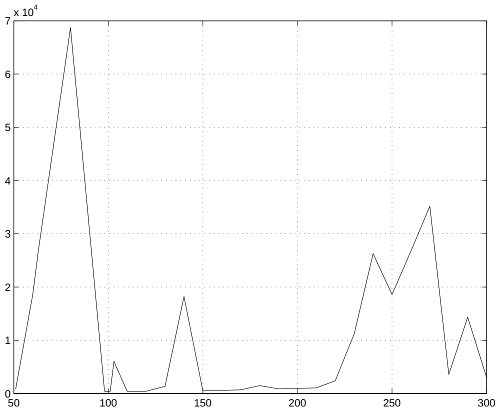

Fig. 1. Optimal value of the cost function versus number of sampling points N, for controller *SOF*2.

As for the dynamic controllers *DOF*1 and *DOF*2, it is apparent from the results summarised in Table 1 that the use of a Kalman filter ensures far better performance with respect to a simple pseudo-differentiator for the estimation of the unmeasurable angular rate. Eventually, the *DOF*2 design provides highly satisfactorily performance in terms of both stability degree and cost function.

In order to illustrate the time domain behavior of the fixed structure controllers, some simulation examples are presented, showing the transient following a (small) initial perturbation of the attitude dynamics with respect to the nominal Earth pointing equilibrium.

It must be noted that, whilst all the proposed control laws achieve comparable levels of performance in terms of spacecraft attitude control, looking at Figures 2 through 9 it can be easily inferred that the *DOF*2 design alone produces a solution affected by undesirable noise. This is not due to design deficiencies, but to the fact that this was the only control law simulated in presence of a noisy measurement feedback. Moreover, Figures 2 through 5 show that, as was expected, while increasing the number of sampling points both accuracy increases and transient durations decrease. Figures 6 and 7 show the loss of pointing accuracy and the arising of high frequency components in the control action for the *SOF*2 design as a consequence of the adoption of a non optimal sampling time Δ. Finally, Figures 8 and 9 show the performance achievable via ad hoc tuned dynamic output feedback *DOF*1, *DOF*2 control laws.

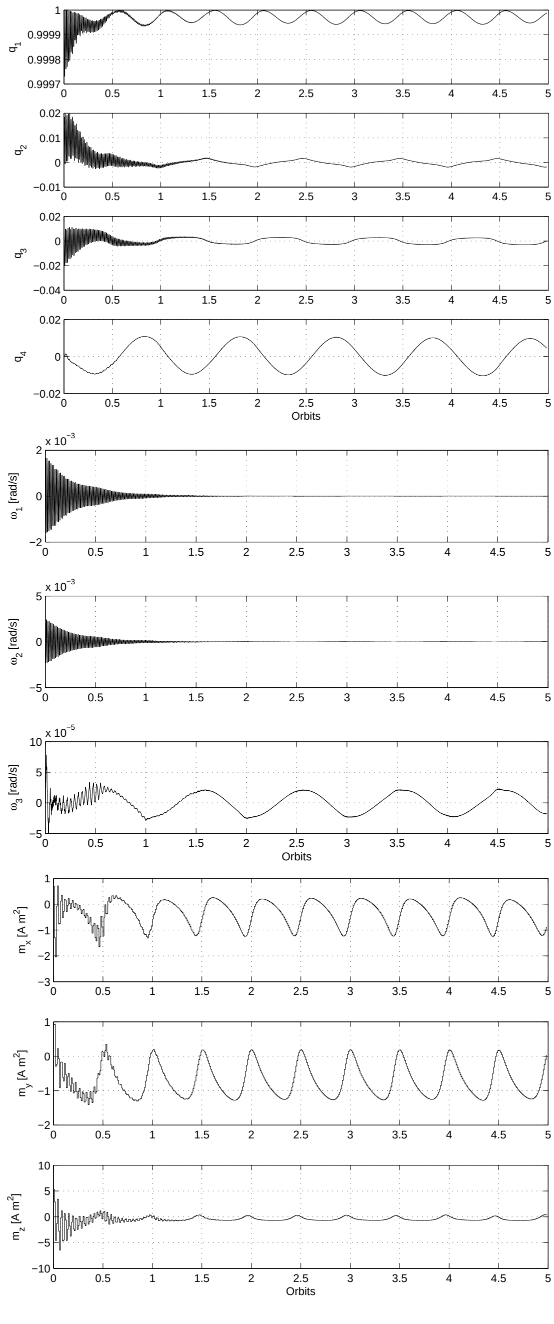

Fig. 2. Quaternion, angular rates and control dipole moments: static state feedback controller SSF ($N = 100$).

5. CONCLUSIONS

In this paper the problem of designing discrete-time attitude controllers for magnetically actuated spacecrafts has been considered. Several methods have been discussed and a novel approach to the tuning of various classes of "projection based" controllers has been proposed, relying on periodic optimal output feedback control techniques. The performances of the proposed control algorithms have been discussed and illustrated in a detailed simulation study, where an Earth pointing spacecraft operates in a near polar orbit (87^o inclination) with an altitude of 450

	stability degree ($N = 100$)	cost function ($N = 100$)	stability degree ($N = 300$)	cost function ($N = 300$)
open loop	1.0243e+004			
PSF	2.2812e-002	5.89e+001	1.017e-004	1.3162e+002
SSF	5.3867e-002	6.441e+001	3.889e-003	1.3830e+002
*SOF*1	5.9256e-002	7.6545e+001	2.3771e-003	1.6847e+002
*SOF*2	2.9372e-002	3.8538e+002	3.4939e-001	3.0298e+003
*DOF*1	2.0356e-002	4.2717e+002	3.4938e-001	3.0298e+003
*DOF*2	5.3867e-002	6.441e+001	3.889e-003	1.3830e+002

Table 1. Open and closed loop stability degree and cost function for $N = 100$ and $N = 300$.

Km, an orbital period of 5614.8 seconds and (possibly) partial information availability upon the system state.

All the considered control designs have provided highly satisfactorily performances, and proved the capability to overcome one or both the main restrictions posed by the reference periodic optimal state feedback control design, *i.e.*, demanding memory storage requirements and full state measurements availability. Surprisingly enough, for the considered simulation study the spacecraft has been showed to be controllable even in absence of angular rates feedback, *i.e.*, in a pure "positional" feedback framework. While in the continuous-time domain such a control strategy would fail in damping the spacecraft angular motion, an artificial damping is now introduced in the system throughout the sampling performed on data.

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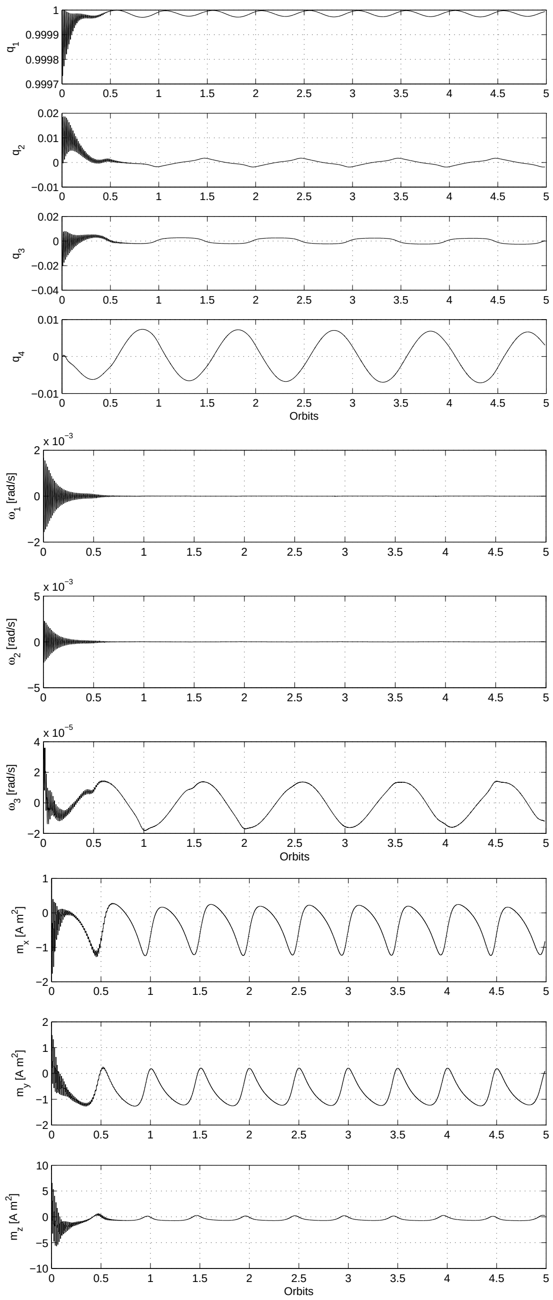

Fig. 3. Quaternion, angular rates and control dipole moments: static state feedback controller SSF ($N = 300$).

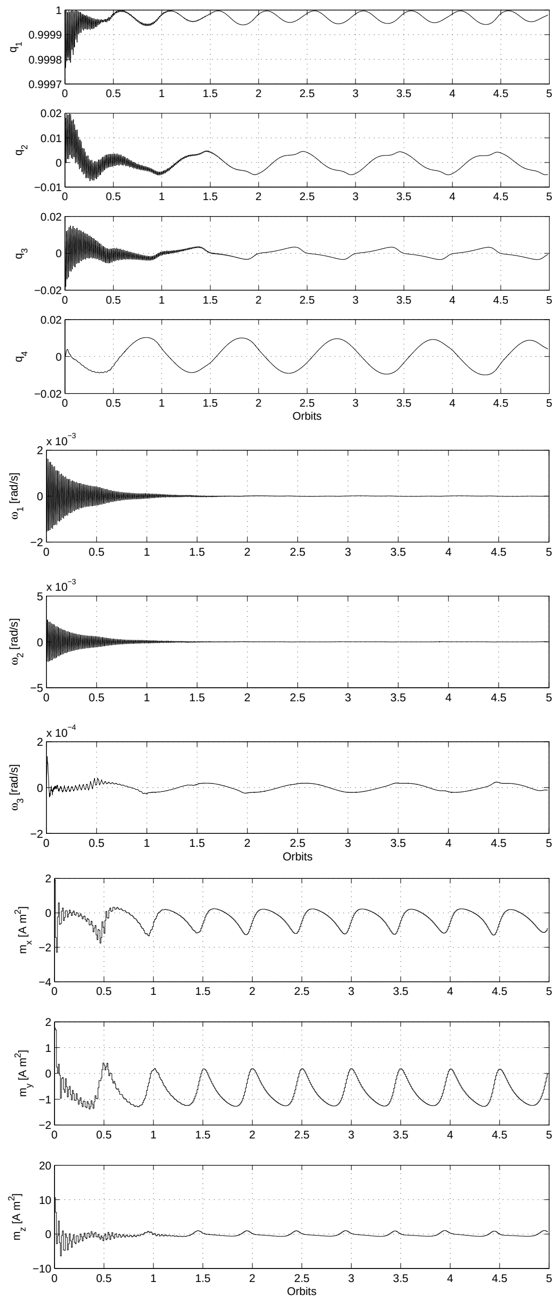

Fig. 4. Quaternion, angular rates and control dipole moments: static output feedback SOF1 ($N = 100$).

Appendix A. DIGITAL CONTROL OF THE HARMONIC OSCILLATOR

The following example has been taken from (Kabamba, 1987), and has been reproduced here to provide an insight of the unexpected good results obtained for the $SOF2$ design.

Consider the following system: $n = 2$ and $m = p = 1$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{A.1}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{A.2}$$

where the system states x_1 and x_2 represent the position and velocity of an undamped harmonic oscillator, u is the input force and y is the position measurement respectively. The system is not asymptotically stable and we wish to stabilize it by feedback.

Suppose we use continuous time direct output feedback of the form $u(t) = p(t)y(t)$. The closed loop state equation becomes

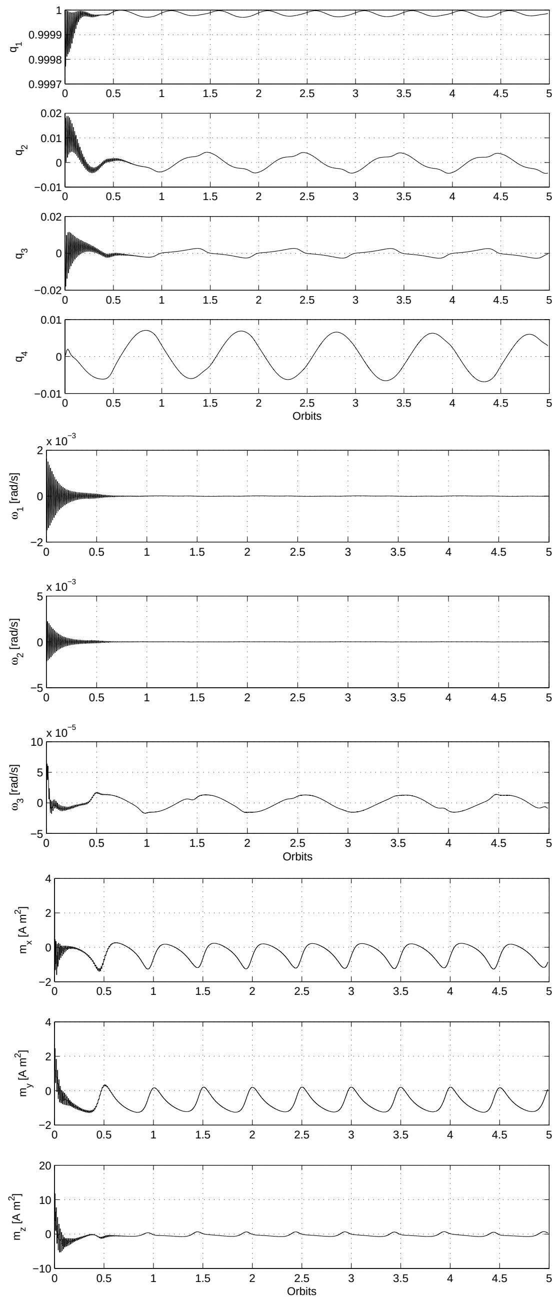

Fig. 5. Quaternion, angular rates and control dipole moments: static output feedback SOF1 ($N = 300$).

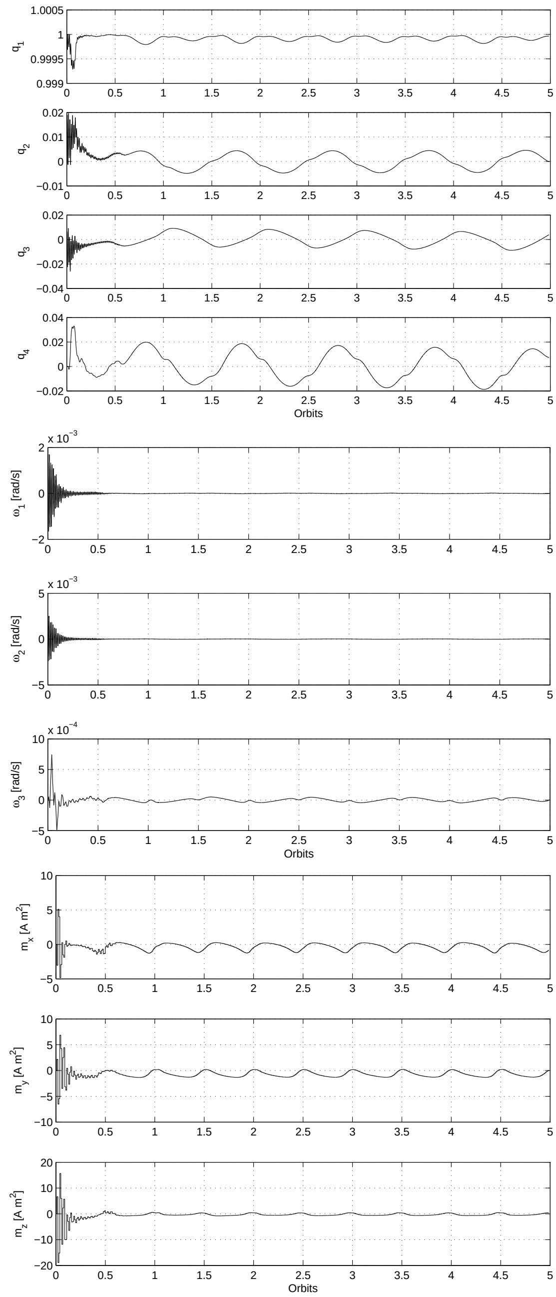

Fig. 6. Quaternion, angular rates and control dipole moments: static output feedback SOF2 ($N = 100$).

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 + p(t) & 0 \end{bmatrix} x \tag{A.3}$$

The trace of the state matrix of (A.3) is zero; therefore, by Jacobi-Liouville's theorem (Brockett, 1970), the determinant of the state transition matrix of (A.3) will always be 1, regardless of the time history of $p(t)$. As a consequence, *system* (A.1)-(A.2) *cannot be made asymptotically stable by continuous time direct output feedback*. This reflects the fact that, with a pure positional feedback, it is only possible to shift the natural frequency of the harmonic oscillator to a desired value, but not to introduce any damping in the system.

Anyway, if a digital control with a zero order hold of the form $u(t) = p\,y(kT)$, $t \in [kT,\,(k+1)T]$, k integer and sampling time T is adopted, the system (A.1), (A.2) can be made asymptotically stable. The discrete-time system is given by

$$u(t) = p\,y_k$$

$$x_{k+1} = e^{AT} x_k + \int_0^T e^{A(T-\tau)} B u(\tau) d\tau = \Phi_k x_k + \Gamma_k u_k \tag{A.4}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \tag{A.5}$$

where

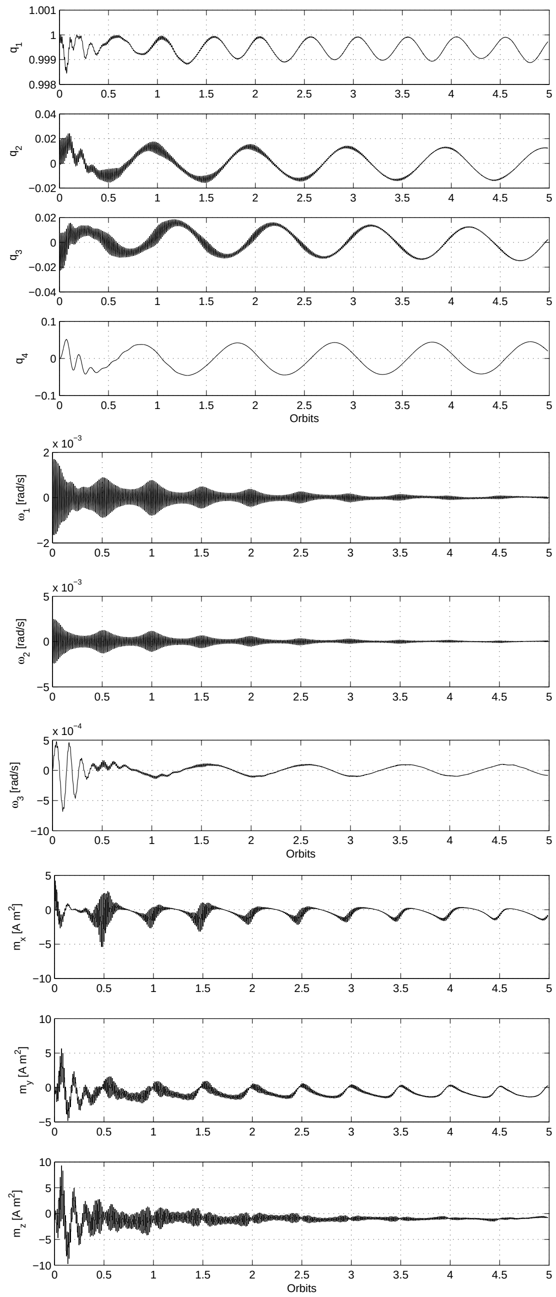

Fig. 7. Quaternion, angular rates and control dipole moments: static output feedback SOF2 ($N = 300$).

$$\Phi_k = \begin{bmatrix} cos(T) & sin(T) \\ -sin(T) & cos(T) \end{bmatrix} \qquad \Gamma_k = \begin{bmatrix} 1 - cos(T) \\ sin(T) \end{bmatrix}$$

with closed loop discrete state equation in the form

$$x_{k+1} = \begin{bmatrix} (1-p)\, cos(T) + p & sin(T) \\ (p-1)\, sin(T) & cos(T) \end{bmatrix} x_k \qquad \text{(A.6)}$$

and characteristic polynomial

$$z^2 - [2\, cos(T) + p\, (1 - cos(T))] z + 1 + p\, (cos(T) - 1) = 0$$

A simple analysis indicates that asymptotically stability is attained *iff* a gain $p \in (0, 1)$ is chosen, irrespective of the adopted sampling time *T*. Anyway, even for this simple example, the closed loop eigenvalues cannot be assigned arbitrarily. Their modulus have to lie on the sharp surfaces depicted in Figure A.1, depending upon the chosen values of the sampling time T and feedback gain p.

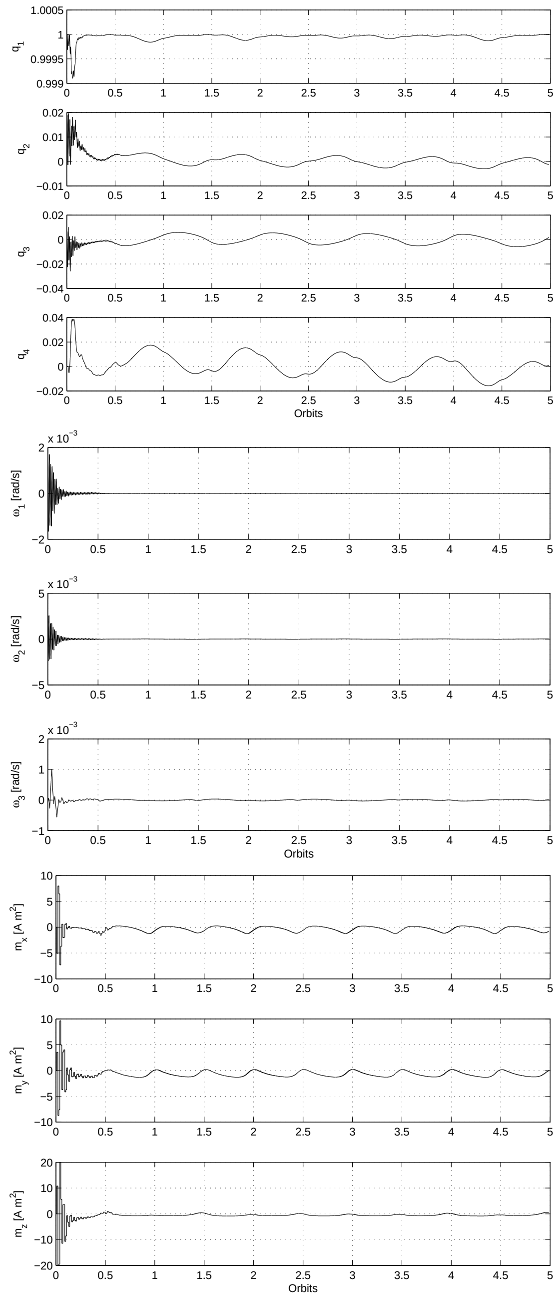

Fig. 8. Quaternion, angular rates and control dipole moments: DOF1 ($N = 100$).

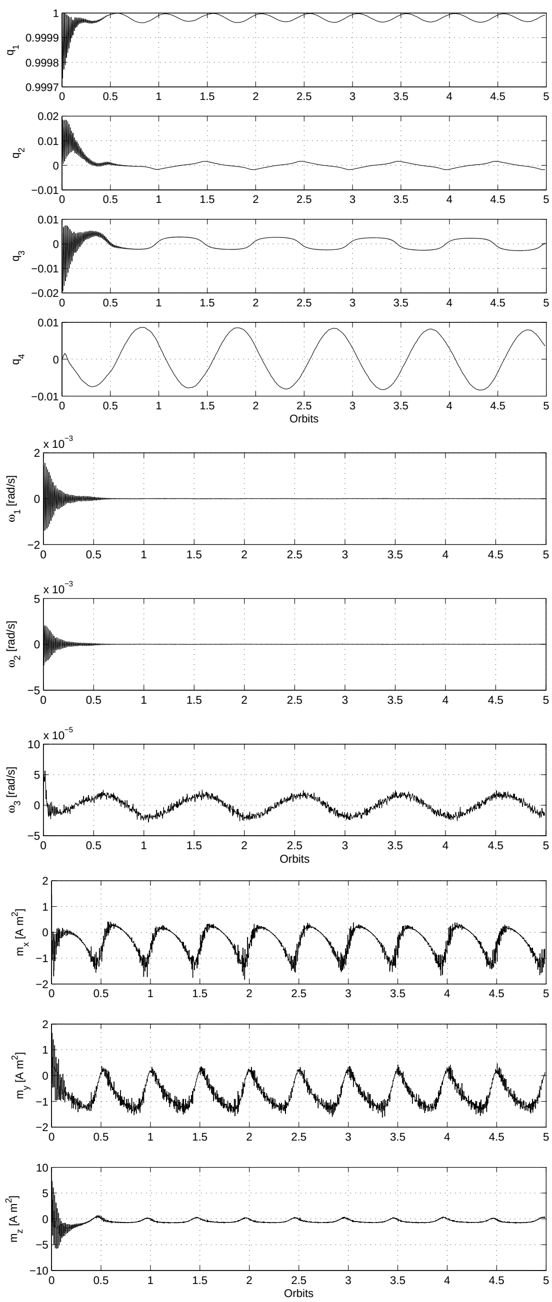

Fig. 9. Quaternion, angular rates and control dipole moments: DOF2 ($N = 300$).

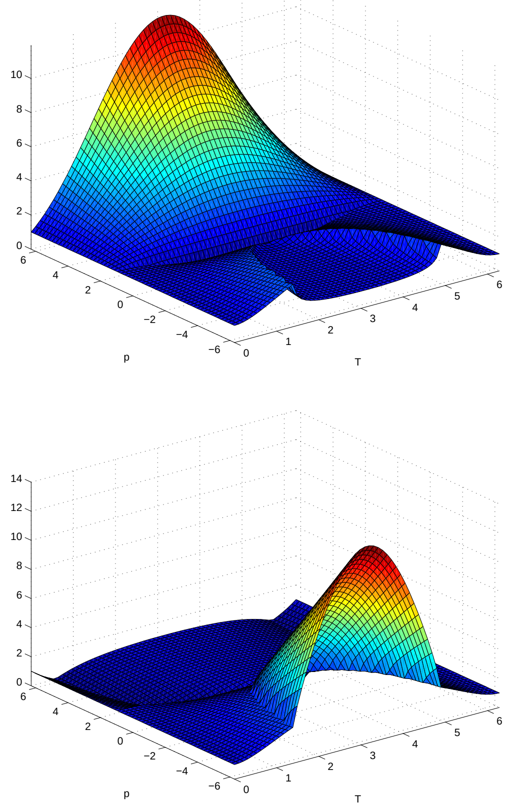

Fig. A.1. Eigenvalues modulus vs feedback gain p and sampling time T for the harmonic oscillator.