On the Geršgorin Disc Theorem applied to Radar Polarimetry

A. Danklmayer (1), W.-M. Boerner (2), M. Chandra (3)
(1) Microwaves and Radar Institute, DLR – German Aerospace Centre, Oberpfaffenhofen, Germany
(2) University of Illinois at Chicago, Department of Electrical and Computer Engineering, Communications, Sensing & Imaging and Navigation Laboratory, Chicago IL/USA
(3) Chair for Microwave Engineering and Photonics, Chemnitz University of Technology, Germany

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Abstract

This contribution is concerned with the mathematical formulation and theoretical background of the Geršgorin discs in the context of Radar Polarimetry. We consider strict radar backscattering, the monostatic case, characterised by the random Sinclair matrix S(t) in a common linear basis. Using the target feature vectors leads to the Hermitian positive semidefinite Covariance matrices, where the eigenvalues are obtained by unitarily diagonalization. A special region G(A), called the Geršgorin discs and the associated boundaries denoted by Geršgorin circles is considered to be of possible value in revealing information about the eigenvalues of a given Covariance matrices. We arrive at particular classes of easily computed regions in the plane that are guaranteed to include the eigenvalues of a given covariance matrix.

1 Introduction

We consider strict radar backscattering (the monostatic case) characterized by the random Sinclair matrix $S(t)$ in the common linear $\{x, y\}$- basis

$$S(t) = \begin{bmatrix} S_{xx}(t) & S_{xy}(t) \\ S_{yx}(t) & S_{yy}(t) \end{bmatrix}, \tag{1}$$

In the case of reciprocal backscattering the Sinclair matrix is symmetric $S_{xy} = S_{yx}$ for a deterministic or point target and $S_{xy}(t) = S_{yx}(t)$ for any instant of time or space for a reciprocal random target. A change of the orthonormal polarization basis induces a unitary consimilarity transformation for $S(t)$.

$$S(t) \to S'(t) = U^T S(t) U, \tag{2}$$

This implies that the Sinclair matrix $S(t)$ due to its symmetry can be condiagonalized for any instant of time by unitary consimilarity with the unitary matrix $U(t)$. This follows from Takagi's theorem. There is, however, a unique unitary matrix only for point targets with a delta-type probability density function. We consider the backscatter case and omit the subscript. The standard target feature vector in the general case are given by

$$\vec{k}_4(t) = \operatorname{vec} S(t) = \begin{bmatrix} S_{xx}(t) \\ S_{yx}(t) \\ S_{xy}(t) \\ S_{yy}(t) \end{bmatrix} \tag{3}$$

The corresponding covariance matrices are given by

$$C_4 = < \vec{k}_4(t) \vec{k}_4^{\dagger} > \tag{4}$$

The covariance matrices are Hermitian positive semidefinite and can be diagonalized by general unitary similarity transformations with a certain 4×4 unitary matrice V

$$V^{-1} C_4 V = \Lambda_4 = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}, \tag{5}$$

$$C_4 = V \Lambda_4 V^{-1} \quad \text{with} \quad 0 \le \lambda_4 \le \lambda_3 \le \lambda_2 \le \lambda_1 .$$

With $V = [\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4]$ we obtain the eigenvalue/eigenvector equations

$$C_4 \hat{x}_i = \lambda_i \hat{x}_i \quad \text{with} \quad < \hat{x}_i, \hat{x}_j > = \hat{x}_i^{\dagger} \hat{x}_j = \lambda_i \delta_{ij} \quad (i = 1, 2, 3, 4). \tag{6}$$

All the eigenvectors can be multiplied by arbitrary phase factors $\hat{x}_i \to \exp(j\phi_i)\hat{x}_i$. If all four eigenvalues are different there are four one-dimensional C_4 invariant subspaces: Span($\hat{x}_i$), $i = 1, ..., 4$. The total number of invariant subspaces (including the zero subspace and the entire space C^4) is $2^4 = 8$. These subspaces assumes a particularly simple form if the unitary similarity to the diagonal form Λ_4 is used. Then

$$\hat{x}_i = \hat{e}_i \quad \rightarrow \quad \Lambda_4 \hat{e}_i = \lambda_i \hat{e}_i \quad (i = 1, \ldots, 4). \tag{7}$$

For backscattering the space $\mathbb{C}^4$ containing the general vectors $\vec{k}_4(t)$ is restricted to the subspace C_s^4 spanned by the vectors $\vec{k}_4^{(s)}(t)$ with $S_{xy}(t) = S_{yx}(t)$. For the covariance matrix this can be expressed in the form

$$C_s^4 = PC^4 = \operatorname{Im} P \quad \text{with}$$
$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{8}$$

where P is a projector $P^2 = P$. The projector P can be expressed in the following way:

$$P = B^+ B \quad \text{with} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

$$\text{and} \quad B^+ = B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

B is a 3×4 matrix and hence has no inverse in the ordinary sense, The matrix B^+ is the so-called Moore-Penrose inverse of B and is characterized as the solution of the following equations

$$BB^+B = B \quad \text{and} \quad B^+BB^+ = B^+. \tag{11}$$

Note that $BB^+ = I_3$, the 3×3 unit matrix.

The operator B is a transformation from $\mathbb{C}^4 \rightarrow \operatorname{Im} P$ with the properties

$$\operatorname{Im} B = \begin{cases} \operatorname{Im} P & \text{if } \vec{x} \in \operatorname{Im} P \\ 0 & \text{if } \vec{x} \in \operatorname{Ker} P \end{cases}. \tag{12}$$

From the general bi-static scattering matrix C_4 we obtain for strict backscattering the singular matrix given by (14)

This matrix can be decomposed as

$$C_{4,b} = \operatorname{Re} C_{4,b} + j \operatorname{Im} C_{4,b} \tag{13}$$

where $\operatorname{Re} C_{4,b}$ is given by (15) and $\operatorname{Im} C_{4,b}$ by (16). $\operatorname{Re} C_{4,b}$ is symmetric and $\operatorname{Im} C_{4,b}$ skew-symmetric.

$$C_{4b} = \begin{bmatrix} <|S_{xx}(t)|^2> & <S_{xx}(t)S_{xy}^*(t)> & <S_{xx}(t)S_{xy}^*(t)> & <S_{xx}(t)S_{yy}^*(t)> \\ <S_{xy}(t)S_{xx}^*(t)> & <|S_{xy}(t)|^2> & <|S_{xy}(t)|^2> & <S_{xy}(t)S_{yy}^*(t)> \\ <S_{xy}(t)S_{xx}^*(t)> & <|S_{xy}(t)|^2> & <|S_{xy}(t)|^2> & <S_{xy}(t)S_{yy}^*(t)> \\ <S_{yy}(t)S_{xx}^*(t)> & <S_{yy}(t)S_{xy}^*(t)> & <S_{yy}(t)S_{xy}^*(t)> & <|S_{yy}(t)|^2> \end{bmatrix}. \tag{14}$$

$$\operatorname{Re} C_{4b} = \begin{bmatrix} <|S_{xx}(t)|^2> & \operatorname{Re}<S_{xx}(t)S_{xy}^*(t)> & \operatorname{Re}<S_{xx}(t)S_{xy}^*(t)> & \operatorname{Re}<S_{xx}(t)S_{yy}^*(t)> \\ \operatorname{Re}<S_{xx}(t)S_{xy}^*(t)> & <|S_{xy}(t)|^2> & <|S_{xy}(t)|^2> & \operatorname{Re}<S_{xy}(t)S_{yy}^*(t)> \\ \operatorname{Re}<S_{xx}(t)S_{xy}^*(t)> & <|S_{xy}(t)|^2> & <|S_{xy}(t)|^2> & \operatorname{Re}<S_{xy}(t)S_{yy}^*(t)> \\ \operatorname{Re}<S_{xx}(t)S_{yy}^*(t)> & \operatorname{Re}<S_{xy}(t)S_{yy}^*(t)> & \operatorname{Re}<S_{xy}(t)S_{yy}^*(t)> & <|S_{yy}(t)|^2> \end{bmatrix} \tag{15}$$

$$\operatorname{Im} C_{4b} = \begin{bmatrix} 0 & \operatorname{Im}<S_{xx}(t)S_{xy}^*(t)> & \operatorname{Im}<S_{xx}(t)S_{xy}^*(t)> & \operatorname{Im}<S_{xx}(t)S_{yy}^*(t)> \\ -\operatorname{Im}<S_{xx}(t)S_{xy}^*(t)> & 0 & 0 & \operatorname{Im}<S_{xy}(t)S_{yy}^*(t)> \\ -\operatorname{Im}<S_{xx}(t)S_{xy}^*(t)> & 0 & 0 & \operatorname{Im}<S_{xy}(t)S_{yy}^*(t)> \\ -\operatorname{Im}<S_{xx}(t)S_{yy}^*(t)> & -\operatorname{Im}<S_{xy}(t)S_{yy}^*(t)> & -\operatorname{Im}<S_{xy}(t)S_{yy}^*(t)> & 0 \end{bmatrix} \tag{16}$$

This matrix operator acts in the restricted space C_s^4 which is invariant with respect to the projector P. Hence we can write

$$\begin{aligned} C_{4b} &= PC_{4b}P = B^+BC_{4b}B^+B =: B^+C_3B \\ \text{with} \quad C_3 &= BC_{4b}B^+ \end{aligned} \tag{17}$$

or explicitly

$$C_3 = \begin{bmatrix} <|S_{xx}(t)|^2> & \sqrt{2}<S_{xx}(t)S_{xy}^*(t)> & <S_{xx}(t)S_{yy}^*(t)> \\ \sqrt{2}<S_{xy}(t)S_{xx}^*(t)> & 2<|S_{xy}(t)|^2> & \sqrt{2}<S_{xy}(t)S_{yy}^*(t)> \\ <S_{yy}(t)S_{xx}^*(t)> & \sqrt{2}<S_{yy}(t)S_{xy}^*(t)> & <|S_{yy}(t)|^2> \end{bmatrix}. \tag{18}$$

Being a similarity transformation the matrices C_{4b} and $\tilde{C}_{4b}$ have the same eigenvalues and the matrix $\tilde{C}_{4b}$ h is also Hermitian positive semidefinite. Deflation can be performed in any basis of the target feature vector.

The 3×3 covariance matrix C_3 can thus be generated directly from the feature vector

$$\vec{k}_3(t) = B\vec{k}_{3,b}(t) = \begin{bmatrix} S_{xx}(t) \\ \sqrt{2}\, S_o(t) \\ S_{yy}(t) \end{bmatrix} \quad \text{with} \quad S_o(t) = S_{xy}(t) = S_{yx}(t). \tag{19}$$

by the standard definition

$$\begin{aligned} C_3 &= < B\vec{k}_{4,b}(t)\vec{k}_{4,b}^{\dagger}(t)B^T > \\ &= < \vec{k}_3(t)\vec{k}_3^{\dagger}(t) > . \end{aligned} \tag{20}$$

The unitary matrix $U^T \otimes U^T$ has the form

$$W^{\dagger} = (U \otimes U)^T = U^T \otimes U^T = \begin{bmatrix} u_{11}u_{11} & u_{11}u_{21} & u_{21}u_{11} & u_{21}u_{21} \\ u_{11}u_{12} & u_{11}u_{22} & u_{21}u_{12} & u_{21}u_{22} \\ u_{12}u_{11} & u_{12}u_{21} & u_{22}u_{11} & u_{22}u_{21} \\ u_{12}u_{12} & u_{12}u_{22} & u_{22}u_{12} & u_{22}u_{22} \end{bmatrix} \tag{21}$$

and if applied to a vector

$$\vec{x} = \begin{bmatrix} a \\ b \\ b \\ c \end{bmatrix} \in C_s^4 = PC^4 = \operatorname{Im} P \quad \Rightarrow \tag{22}$$

$$\vec{x}\,' = W^{\dagger}\vec{x} = U^T \otimes U^T\vec{x} = \begin{bmatrix} a' \\ b' \\ b' \\ c' \end{bmatrix} \in C_s^4, \tag{23}$$

This in particular applies to the standard target feature vector $\vec{k}_{4,b}$,

i.e., the subspace C_s^4 is invariant under the unitary transformation $U^T \otimes U^T$.

In general the unitary transformations that diagonalize the covariance matrices are not of the form of a polarimetric basis transformation, i.e., in general

$$\begin{aligned} U(C_S) &\neq (U(S) \otimes U(S))^T \quad \text{and} \\ U(C_J) &\neq (U(C_J) \otimes U^*(C_J))^T . \end{aligned} \tag{24}$$

In the following we refer to some results contained in Horn and Johnson [1] and Varga [2]

2 Geršgorin disc Theorem

Let $[a_{ij}] \in M_n$, and

$$R_i'(A) \equiv \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad 1 \le i \le n \tag{25}$$

denote the *deleted absolute row sums* of A. Then all the eigenvalues of A are located in the union of n discs

$$\bigcup_{i=1}^{n} \{z \in C : |z - a_{ii}| \le R_i'(A)\} \equiv G(A). \tag{26}$$

Furthermore, if a union of k of these n forms a connected region that is disjoint from all the remaining $n-k$ discs then there are precisely k eigenvalues of A in this region.

The region $G(A)$ if often called the *Geršgorin region* (for rows) of A; the individual discs in $G(A)$ are called *Geršgorin discs*, and the boundaries of these discs are called *Geršgorin circles*. Since the matrices A and A^T have the same eigenvalues, one can obtain a Geršgorin disc theorem for columns by applying the Geršgorin disc theorem to A^T to obtain a region that contains the eigenvalues of A and is specified in terms of deleted absolute column sums

$$C_j'(A) \equiv \sum_{\substack{i=1 \\ i\neq j}}^{n} |a_{ij}|, \quad 1 \le j \le n. \tag{27}$$

Corollary. If $A = [a_{ij}] \in M_n$, then all the eigenvalues of A are located in the union of n discs

$$\bigcup_{j=1}^{n}\{z \in C : |z - a_{jj}| \le C_j^{'}\} = G(A^T). \quad (28)$$

All eigenvalues of A lie in the intersection $G(A) \cap G(A^T)$.

In the following we understand by C covariance matrix C_S. With the antenna transform U_2 and $C_a \equiv C_a(U_2) = (U_2 \otimes U_2)^T C (U_2 \otimes U_2)^*$ we have the diagonal part of any conceivable C

$$D(C_a) = \begin{bmatrix} D_{xx}(C_a) & 0 & 0 & 0 \\ 0 & D_{yx}(C_a) & 0 & 0 \\ 0 & 0 & D_{xy}(C_a) & 0 \\ 0 & 0 & 0 & D_{yy}(C_a) \end{bmatrix}. \quad (29)$$

The diagonal elements of $C_p = U_4^{-1} C U_4$ for any unitary U_4 matrix form the diagonal matrix

$$D(C_p) = \begin{bmatrix} D_{xx}(C_p) & 0 & 0 & 0 \\ 0 & D_{yx}(C_p) & 0 & 0 \\ 0 & 0 & D_{xy}(C_p) & 0 \\ 0 & 0 & 0 & D_{yy}(C_p) \end{bmatrix}. \quad (30)$$

In both cases for any matrix C_a and C_p we have for the sum of diagonal elements

$$\mathrm{trace}(C_a) = \mathrm{trace}(C_p) = \sum_{i=1}^{4} \lambda_i. \quad (31)$$

This is the trace invariance of unitary transformations.

We introduce

$$R_i(U_2) = \sum_{j=1,\, j\neq i}^{4} |\, C_a(U_2)\,|_{ij}|, \quad (i = 1, 2, 3, 4). \quad (32)$$

The matrices U_2 will be given in the form

$$U_2 \equiv U_2(\rho) = \frac{1}{\sqrt{1 + |\rho|^2}} \begin{bmatrix} 1 & -\rho^* \\ \rho & 1 \end{bmatrix} \quad \text{with} \quad \det(U_2) = 1. \quad (33)$$

where ρ is the complex polarization ratio. Then

$$U_2 \otimes U_2 = \frac{1}{1 + |\rho|^2} \begin{bmatrix} 1 & -\rho^* & -\rho^* & \rho^{2*} \\ \rho & 1 & -|\rho|^2 & -\rho^* \\ \rho & -|\rho|^2 & 1 & -\rho^* \\ \rho^2 & \rho & \rho & 1 \end{bmatrix} \quad (34)$$

with $\det(U_2 \otimes U_2) = 1$. These matrices form a true subgroup of the group of all unitary 4×4 matrices U_4 and depend only on the complex parameter ρ or on two real parameters for which we take the ellipticity angle ε with $-45^\circ \le \varepsilon \le 45^\circ$ and the orientation angle θ with $-90^\circ < \theta \le 90^\circ$. Since the set of these matrices U_2 is compact and all functions involved are continuous then invoking the Weierstrss theorem the functions $R_i(U_2)$ assume a maximum and a minimum at $\rho_{\max}$ and $\rho_{\min}$, respectively.

Since the set $U_2(\rho) \otimes U_2(\rho)$ is only a subset of U_4 we have in general

$$\min R_i(U_2(\rho)) = R_i(U_2(\rho_{\min})) \ge 0, \quad (i = 1, 2, 3, 4).$$

These minima are best determined numerically for any given covariance matrix.

The precise relationship between the main diagonal elements and the eigenvalues is given by the notion of majorization, see Horn and Johnson [1]

3 Conclusions

The Geršgorin disc theorem is presented and adopted to the covariance matrices used in radar polarimetry, where the theorem shows potential to allow for target identification and classification which has to be further investigated.

4 Literature

[1] Horn, R. A. and Ch. R. Johnson, Matrix Analysis, Cambridge University Press, New York, 1985.

[2] Varga, R. S., Geršgorin and his circles, Springer, 2004

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