

Enhanced Modal Approach for Model Reduction

A. Varga

DLR - Oberpfaffenhofen
German Aerospace Research Establishment
Institute for Robotics and System Dynamics
P.O.B. 1116, D-82230 Wessling, Germany
E-mail: `Andreas.Varga@dlr.de`

Abstract. The general applicability of the modal approach for model reduction is restricted by the lack of guaranteed bounds for approximation errors and the lack of a satisfactory modal dominance analysis procedure. Functional and computational enhancements of this approach are proposed. Functional enhancements arise by combining the modal techniques with other methods and by using improved dominance analysis techniques. The computational enhancements are the results of employing numerically reliable algorithms for both dominance analysis as well as for model reduction.

Keywords. Model reduction, mathematical modelling, computational methods, modal techniques, modal dominance analysis.

1 Introduction

The modal approach to model reduction proposed initially by Davison [4] was later extended with new variants by several authors: Marschall [13], Chidambara [2], Fossard [7], Litz [11], Skelton and Yousuff [16] and others. The importance of the modal approach as a useful model reduction technique resides in its applicability to reduce high order systems as those arising for example from modelling of large mechanical structures or of large power systems. The method can handle models with lightly damped modes and even unstable systems. In case of very large order systems, the modal technique is one of the very few applicable methods.

Several limitations of the modal approach raise problems for its general usability. In the first place, the lack of a generally applicable modal dominance analysis method prevents the use of this method in many cases as for example when the original system has multiple or nearly equal poles. The existing methods fail sometimes even to detect exact structural non-minimality, that is, poles which are uncontrollable or unobservable. Another weakness

of this approach is the lack of a guaranteed bound for the approximation error which has as consequence the frequent need to experiment on a trial and error basis with different approximations.

In this paper we shortly survey the main existent modal reduction approaches and some of available techniques for modal dominance analysis. Then we discuss possible enhancements of the modal reduction approach. These enhancements consist in: 1) combining the modal techniques with other approaches; 2) using a new, more powerful method for modal dominance analysis; and 3) using numerical techniques with guaranteed numerical reliability. The proposed new modal approach is well suited for robust software implementation. An example worked out with a MATLAB implementation of this approach illustrates the effectiveness of the proposed enhancements.

2 Modal Techniques for Model Reduction

Consider the n -th order original state-space model $G := (A, B, C, D)$ with the $p \times m$ transfer-function matrix (TFM)

$$G(\lambda) = C(\lambda I - A)^{-1}B + D \quad (1)$$

and let $G_r := (A_r, B_r, C_r, D_r)$ be an r -th order approximation of the original model ($r < n$), with the TFM

$$G_r = C_r(\lambda I - A_r)^{-1}B_r + D_r. \quad (2)$$

In (1) and (2) λ stays either for the complex variable s in the Laplace-transform or the complex variable z in the Z-transform, depending on the type of the system, continuous-time or discrete-time, respectively.

The modal approach to model reduction can be interpreted as performing a similarity transformation Z on the system matrices yielding

$$\left[\begin{array}{c|c} Z^{-1}AZ & Z^{-1}B \\ \hline CZ & D \end{array} \right] := \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D \end{array} \right], \quad (3)$$

where A_1 and A_2 contains the r dominant and respectively, the $n - r$ non-dominant eigenvalues (modes) of A , and then defining the reduced model on the basis of this partitioned representation. The above partition of system matrices is equivalent with the additive decomposition $G = G_1 + G_2$, where $G_1 := (A_1, B_1, C_1, D)$ and $G_2 := (A_2, B_2, C_2, 0)$ are the dominant and non-dominant subsystems, respectively.

We consider three basic modal approaches for model reduction which encompass the main characteristics of several existing modal techniques. In order to simplify and unify the presentation, we reinterpreted the existing approaches in terms of input-output maps (TFMs). For the purpose of our discussion we assume that the original system is already additively decomposed. Furthermore we assume that the system is asymptotically stable. This

latter assumption is only a technical one, because for an unstable system the modal approach can be performed on its stable projection.

Method 1. Define $G_r := (A_1, B_1, C_1, D)$. This is basically the modal approximation proposed in [4]. The approximation error $\Delta = G - G_r$ tends to zero at high frequencies. However, the DC-gains mismatch of the original and reduced models could be large. \square

Method 2. Define $G_r := (A_1, B_1, C_1, D + G_2(\gamma))$, where $\gamma = 0$ for a continuous-time system and $\gamma = 1$ for a discrete-time system. Note that $G(\gamma) = G_r(\gamma)$, and thus this approximation preserves the DC-gain of the original system, but the approximation error at high frequencies could be large. The methods of [13, 2, 7], which compensate the steady-state errors can be viewed as particular cases of Method 2. \square

Method 3. Define $G_r := (A_1, B_1, C_1 + C_2E, D)$, where E is to be determined such that $G(\gamma) = G_r(\gamma)$. This approximation automatically ensures small errors at high frequencies. From the equality of DC-gains follows that E should satisfy

$$C_2(\gamma I - A_2)^{-1}B_2 = C_2E(\gamma I - A_1)^{-1}B_1.$$

This is a system of pm linear equations with $(n-r)r$ unknowns and a solution (with possibly minimal norm) generically exists provided $pm \leq (n-r)r$, a condition fulfilled in most applications. The method of [11] results if we impose the stronger condition

$$(\gamma I - A_2)^{-1}B_2 = E(\gamma I - A_1)^{-1}B_1,$$

which usually leads to an E with higher norm. The generic solvability condition in this case is $m \leq r$, which in most applications is also fulfilled. The additional freedom arising from the non-uniqueness of E can be used to optimally tune the free parameters of E to minimize for instance the output error norm. \square

One difficulty in using the modal approach is the lack of an *a priori* computable bound for the resulting approximation error $\Delta = G - G_r$. The actual error can be computed only after that a choice of the order has been made. Thus model reduction based on the modal approach is done typically on a *trial and error* basis. In contrast, methods based on balancing, as for example the *balance & truncate* (B&T) method [14, 8], provide *a priori* information (the Hankel-singular values) which can be used to select the appropriate order for an acceptable approximation error.

It is possible to combine the modal approach with other techniques. For example, if the system is already decomposed as in (3), then the reduction can be performed separately on G_1 and G_2 . Let $G_r = G_{1r} + G_{2r}$ be the resulting reduced model, where G_{1r} and G_{2r} are the resulting reduced subsystems computed say with the B&T method. If for the separate reduction of terms we have that $\|G_i - G_{ir}\| \leq \varepsilon_i$ for $i = 1, 2$, then $\|G - G_r\| \leq \varepsilon_1 + \varepsilon_2$. Thus, by reducing individually the terms, we can also control the resulting global error

by choosing appropriate orders for the reduced subsystems. The technique can be readily extended to additive decompositions with more than two terms (see section 4) and many variations of it are possible by employing alternative model reduction methods.

The real advantage of such combinations is more evident when we have to reduce very large order models, as those which typically result from finite-element analysis of large mechanical structures. Because of large orders of such models, the modal approach is frequently the only method which can be used for order reduction. This reduction is often only a preliminary step which makes tractable further reductions with the help of more powerful methods.

3 Modal Dominance Analysis Techniques

The main limitation of the modal approach to model reduction is the lack of a reliable, general purpose method for modal dominance analysis. The existence of such a method is highly questionable because for any of known methods examples can be easily constructed showing their failures in producing useful dominance information. We can see this as a basic limitation of the modal approach which can permanently occur, because often the identified dominant parts have still too large orders and thus further reductions should have recourse to alternative techniques. In this section we discuss the limitations of some of existing dominance analysis techniques and in the next section we propose an alternative approach to overcome them.

Consider the system $G := (A, B, C, D)$ with the state matrix A in a *block-diagonal form* (BDF)

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{ccc|c} A_1 & \cdots & 0 & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_k & B_k \\ \hline C_1 & \cdots & C_k & D \end{array} \right]. \quad (4)$$

This partition of system matrices is equivalent with the additive decomposition

$$G = D + \sum_{i=1}^k G_i,$$

where

$$G_i(\lambda) = C_i(\lambda I - A_i)^{-1} B_i, \quad i = 1, \dots, k.$$

We use this decomposition to present an unifying treatment of modal dominance analysis methods.

The earlier modal reduction methods [4, 13, 2, 7] concerns exclusively with *continuous-time* systems and always assume that A is *diagonalizable*, and thus all blocks in (4) are 1×1 . An eigenvalue (mode) λ_i is called *dominant* (or *slow*) if it is situated not too far from the imaginary axis and

non-dominant (or *fast*) otherwise. In the above mentioned methods the fast modes lying far from the imaginary axis are always neglected, even if they have a substantial contribution to the system dynamics.

A more satisfactory approach was proposed by Litz [12]. As dominance index for an eigenvalue λ_i he used the quantity

$$R_i = \|S_1 G_i(0) S_2\|, \quad (5)$$

where S_1 and S_2 are diagonal output and input scaling matrices, respectively, and $\|F\| := \sum_{i,j} |f_{ij}|$ or $\|F\| := \max_{i,j} |f_{ij}|$. Those eigenvalues having the largest dominance indices are called *dominant* and are retained in the reduced model. In order to evidence weak dynamic interactions, Litz also introduced a somewhat heuristically defined frequency-weighted dominance index. The choice of matrices S_1 and S_2 should reflect the relative importance of different output and input variables. A possible choice for the diagonal elements of these matrices is to take them as the reciprocal of the absolute maximum values of the the corresponding output and input variables. Notice that dominance indices which are essentially equivalent with (5) can be defined by replacing R_i with any TFM norm, as for instance the 2-, ∞ - or Hankel-norm. Each TFM $G_i(\lambda)$ being of the form

$$G_i(\lambda) = \frac{C_i B_i}{\lambda - \lambda_i},$$

the evaluation of these norms can be done by using easily computable explicit formulas for both continuous- and discrete-time systems.

The *component cost analysis* method was introduced for continuous-time systems by Skelton and Yousuff [16] to determine the contribution of the components of the state vector to a quadratic cost functional $V = \text{tr}[XC^T C]$, where X is the controllability gramian satisfying

$$AX + XA^T + BB^T = 0.$$

The cost V can be interpreted as the mean-squared output norm corresponding to a zero mean white noise input with unit intensity. The total cost V can be always decomposed as

$$V = \sum_{i=1}^k V_i$$

where $V_i = \text{tr}[XC^T C]_{ii}$ and $|V_i|$ is the so-called *component cost* associated with the state vector component of the system (4) corresponding to the subsystem (A_i, B_i, C_i) .

If all diagonal blocks in A are 1×1 , then $|V_i|$ is called the *modal cost* associated with the eigenvalue λ_i . In this case it is easy to evaluate V_i explicitly. The X_{ij} element of X can be computed as

$$X_{ij} = -\frac{B_i B_j^*}{\lambda_i + \bar{\lambda}_j}$$

and thus

$$V_i = \sum_{j=1}^n X_{ij} C_j^* C_i = - \sum_{j=1}^n \frac{B_i B_j^* C_j^* C_i}{\lambda_i + \bar{\lambda}_j} = \|G_i\|_2^2 + \Delta_i \quad (6)$$

where

$$\|G_i\|_2^2 = - \frac{B_i B_i^* C_i^* C_i}{2\text{Re}(\lambda_i)}, \quad \Delta_i = - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{B_i B_j^* C_j^* C_i}{\lambda_i + \bar{\lambda}_j}$$

Unfortunately, the cost $|V_i|$ associated with an eigenvalue λ_i depends in general also on other eigenvalues of A because of the nonzero term Δ_i in (6). Therefore, only in those special cases when this term is negligible (as for instance for models of lightly damped space structures [15]) the modal costs $|V_i|$ appropriately characterize the modal dominance of the corresponding eigenvalues. In that cases however, these costs reduce to $\|G_i\|_2^2$ and thus provide essentially the same information as any other norm of G_i .

The main limitation of using all dominance measures discussed above is the requirement for A to be diagonalizable. But even if A is diagonalizable, all these dominance measures are not appropriate for detecting exact or nearby structural non-minimality, as evidenced by the following simple example

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{ccc|c} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -10 & 1 \\ \hline 1 & -1 & 1 & 0 \end{array} \right]. \quad (7)$$

Apparently the *slow* eigenvalues $\lambda_1 = \lambda_2 = -1$ should be kept in the reduced model and the *fast* eigenvalue $\lambda_3 = -10$ should be removed. Both the dominance indices $R_1 = R_2 = 1$, $R_3 = 0.1$ computed with (5) and the modal costs $V_1 = V_2 = 1/11$, $V_3 = 1/20$ computed with (6) support this decision. However, it is easy to see that an *exact* minimal realization of this system is

$$\tilde{A} = -10, \quad \tilde{B} = 1, \quad \tilde{C} = 1, \quad \tilde{D} = 0.$$

In the next section we propose a new technique which allows an easy handling of systems with multiple poles or of systems which are exactly or nearly non-minimal.

4 Enhanced Modal Dominance Analysis

The enhancements of the modal dominance analysis which we propose are primarily directed towards handling the cases of multiple eigenvalues and of exact or nearby non-minimality. However, as it will be illustrated by an example, the proposed new technique is also suitable to handle problems with nearly equal dominance measures. In what follows, we assume that in the BDF (4) any of two diagonal blocks have no common eigenvalues. Let

n_i be the order of the i -th block and let $\sigma_j^{(i)}$, $j = 1, \dots, n_i$ the decreasingly ordered *Hankel singular values* (HSV) of the subsystem $G_i := (A_i, B_i, C_i)$. Recall that the HSV are defined as [8]

$$\sigma_j^{(i)} = [\lambda_j(P_i Q_i)]^{\frac{1}{2}}, \quad j = 1, \dots, n_i$$

where P_i and Q_i are the controllability and observability gramians of the system G_i . The highest singular value $\sigma_1^{(i)}$ is the Hankel-norm of the subsystem G_i . The two gramians satisfy appropriate continuous- or discrete-time Lyapunov equations depending on the type of the system.

The usefulness of the HSV in defining dominance measures lies in the following interpretation of singular values: $\sigma_j^{(i)}$ is a measure of the *nearness* of the given subsystem G_i to a j -th order approximation [8]. Based on this interpretation we can introduce the following new dominance concept.

Definition. Let $\varepsilon \geq 0$ be a given threshold value. If $\sigma_j^{(i)} > \varepsilon$ for $j = 1, \dots, r_i$ and $\sigma_j^{(i)} \leq \varepsilon$ for $j = r_i + 1, \dots, n_i$, then r_i eigenvalues of A_i are *dominant* and $n_i - r_i$ are *non-dominant*. \square

We apply this definition to the example in (7) where we consider this time A with two diagonal blocks, the first block is 2×2 with two equal eigenvalues at -1 and the second block is 1×1 containing the eigenvalue -10 . The singular values corresponding to the first block of A are zero. Thus the first two repeated eigenvalues are non-dominant and can be removed from the model.

The new dominance concept differs from the previously discussed dominance measures in several aspects. First, it covers the case of arbitrary order diagonal blocks, and in particular of blocks having equal or nearby eigenvalues. Complex eigenvalues raise no problem because they enter always in complex conjugated pairs in the same block. Apparently the case of larger blocks can be also covered by the *component costs* introduced in [16]. Actually, each component cost characterizes the dominance of a whole block and thus can be seen similar with a norm based dominance measure. It is clear however that such dominance measures are only appropriate for 1×1 blocks. For example, the non-dominant eigenvalues within a larger block can not be detected with the help of the corresponding component cost when at least one eigenvalue of this block has significant contribution to the system dynamics. With the new dominance measure based on HSV the detection of such non-dominant eigenvalues is easy, because always the corresponding HSV are small.

A second aspect is that the new dominance concept offers further support when order reduction is not possible because of the presence of several nearly equally dominant eigenvalues in the model. Even if A is diagonalizable, it is still possible that any of discussed modal dominance measures (including the new one) offer no satisfactory dominance information at the level of 1×1 blocks for a sensible reduction of the order. Because the proposed approach is not related to the diagonalizability of A matrix, it allows to form

larger blocks by grouping together eigenvalues lying in certain regions of the complex plane. Then it is possible to perform a new dominance analysis and model reduction on the resulted enlarged blocks. If the effect of enlarging is still not satisfactory, it is possible to further enlarge the blocks. Successive enlargings lead to a guaranteed right answer, because in the worst case we can work with a single block (the A matrix itself). However, for large models it is likely that a substantial reduction of order occurs with much smaller blocks. This aspect is later illustrated by an example.

A third qualitatively new aspect is that excepting the cases when all eigenvalues are dominant or all are non-dominant, we cannot say explicitly which individual eigenvalues are dominant and which are non-dominant. What we actually know is only the number of dominant eigenvalues r_i in a given block A_i . This information is however sufficient to compute an r_i -th order approximation of G_i lying within a given approximation threshold ε . An additional nice feature of HSV is that for a system having say ℓ uncontrollable and/or unobservable eigenvalues there are exactly ℓ zero singular values in the whole set of HSV. Thus, by setting $\varepsilon = 0$, the dominant eigenvalues are precisely those which are both controllable and observable.

The non-dominant part of an additively decomposed system G can be removed simply by removing successively the non-dominant parts of its additively coupled subsystems G_i , $i = 1, \dots, k$. The reduction of subsystems can be performed by applying one of several existing powerful model reduction methods capable to handle non-minimal systems, as for instance the *balancing-free square-root* variant of B&T method [19]. The following straightforward model reduction procedure represents a general template for an enhanced modal approach based on the new dominance concept.

Model Reduction Procedure Based on Modal Approach.

1. Reduce the system $G = (A, B, C, D)$ to the additively decomposed form (4), where $\Lambda(A_i) \cap \Lambda(A_j) = \emptyset$ for $i \neq j$.
2. For $i = 1, \dots, k$ determine r_i , the number of dominant eigenvalues of the diagonal block A_i .
3. For each n_i -th order subsystem $G_i := (A_i, B_i, C_i)$ compute its r_i -th order dominant part $G_{ir} := (A_{ir}, B_{ir}, C_{ir}, D_{ir})$ by using a suitable model reduction algorithm.
4. Construct $G_r := (A_r, B_r, C_r, D_r)$, where

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array} \right] = \left[\begin{array}{ccc|c} A_{1r} & \cdots & 0 & B_{1r} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_{kr} & B_{kr} \\ \hline C_{1r} & \cdots & C_{kr} & D + \sum_{i=1}^k D_{ir} \end{array} \right]$$

This procedure can be easily implemented to determine a reduced system of a *specified* order r or a reduced system G_r satisfying $\|G - G_r\| \leq \varepsilon_a$, where ε_a is a given absolute error tolerance. In the first case, the orders r_i of the reduced subsystems G_{ir} , $i = 1, \dots, k$ should be selected such that they correspond to the first $r = \sum_{i=1}^k r_i$ largest singular values over the whole collection of $n = \sum_{i=1}^k n_i$ singular values. In the latter case, the orders of reduced subsystems can be usually determined from known error bounding relations. For instance, when using the B&T method we can choose the r_i 's such that for a given ε_a we have

$$\|G - G_r\|_\infty \leq \sum_{i=1}^k \|G_i - G_{ir}\|_\infty \leq 2 \sum_{i=1}^k \sum_{j=r_i+1}^{n_i} \sigma_j^{(i)} \leq \varepsilon_a,$$

where we used the expressions of bounds derived in [8] for the B&T method. Note however that the actual error is generally greater (sometimes possibly much greater) than that resulting from the application of the B&T method directly to the original system. Various other aims (DC-gain matching, phase preserving) can be accommodated by using alternative techniques (see [21] and [6] for recent surveys on model reduction methods). It is easy to see that when A has distinct eigenvalues, then the above procedure can be so devised to be equivalent with any of mentioned modal methods.

The proposed modal approach is particularly well suited for computing minimal state-space realizations of TFMs. In this case, an initial non-minimal realization of a given TFM can be directly constructed in a BDF. If necessary, the required grouping of eigenvalues of A can be obtained by performing simple row and column permutations of matrices A , B and C . In this way, the number of performed operations and the necessary storage are usually less than those required by the minimal realization algorithms based on orthogonal controllability/observability staircase forms [18]. It is also worth to mention (see for details the discussion in [20]) the increased overall computational reliability of such an approach to minimal realization in contrast with the high sensitivity to the choice of zero tolerances of staircase forms based minimal realization algorithms.

5 Numerical Enhancements

The model reduction procedure of previous section can be implemented by using exclusively numerically reliable algorithms. For the computation of the BDF at step 1 the algorithm of [1] can be used followed possibly by the reordering and enlarging of diagonal blocks. Besides simplicity and computational efficiency, the main advantage of this algorithm is its ability to keep under control the condition number of the transformation matrix. This allows the computation of the BDF with a prescribed accuracy loss and thus represents a major numerical enhancement over the diagonalization algorithms based on eigenvector computations. Note however that in many cases (finite-element

models, non-minimal TFM realizations) A is already block-diagonal. In such cases only the reordering of blocks is necessary in order to include nearby eigenvalues in the same blocks.

For the dominance analysis at step 2 the HSV can be computed very accurately by using the *square-root* algorithm of [17]. The same algorithm is applicable to both continuous- and discrete-time systems. The only difference consists in solving continuous- or discrete-time Lyapunov equations to compute the corresponding gramians. The term *square-root* usually designates a class of computational methods with enhanced accuracy in which the computations are based exclusively on square-root information, as for instance in our case the Cholesky factors of the gramians. If $P_i = S_i^T S_i$ and $Q_i = R_i^T R_i$ are the Cholesky factorizations of the controllability and observability gramians, respectively, then the HSV can be computed as the ordinary singular values of the product $S_i R_i^T$. The computation of Cholesky factors can be done by solving directly for these factors the Lyapunov equations satisfied by the gramians [9].

The model reductions at step 3 should be done preferably by using one of the recently developed model reduction algorithms with enhanced accuracy (the so-called *square-root* or *balancing-free square-root* methods) (see [21] for an up-to-date survey). All these methods are appropriate to handle exact or nearby non-minimality and thus can be also used very effectively as minimal realization procedures. Here, the supplementary attribute *balancing-free* designates model reduction methods which, although based on balancing techniques, still avoid the use of possibly ill-conditioned balancing transformations. Instead, the reduced models are computed directly with the help of well conditioned projections, thus enhancing the overall computational accuracy of the model reduction algorithm. Notice that at both steps 2 and 3 additional computational efficiency arises by exploiting the particular quasi-upper triangular form of the diagonal matrices A_i which results usually from the reduction to BDF.

Because of usually low dimensions of subsystems G_i , the involved computational effort is mainly due to the reduction to the BDF and thus is about $15n^3$ operations. If the procedure is properly implemented, all computations can be done practically with minimum additional storage (at most n^2 locations if the reduction to BDF is necessary).

6 Example

We applied the proposed enhanced modal approach to an example given in [10]. The sole purpose of this artificial model was to illustrate the failure of the dominance analysis method proposed by Litz [11]. In spite of that, we feel that this example has also a more generic value because it illustrates a rather common situation of models with clustered modes. Such models appear in many practical model reduction problems of large space structures

[3].

The example is a continuous-time single-input single-output system of order 12. In order to avoid the tedious writing of long coefficients of the numerator (of degree 11) and of the denominator (of degree 12), we give the corresponding transfer function in an additively decomposed form

$$G(s) = -\frac{100}{s+100} + \frac{100s}{s^2+20s+2600} + \frac{32s+640}{s^2+20s+356} + \frac{30s}{s^2+20s+325} + \frac{20s+400}{s^2+20s+200} + \frac{20s+198}{s^2+20s+101} + \frac{10}{s+10}. \quad (8)$$

The system with the transfer function $G(s)$ is stable and minimum-phase, having the following poles and zeros:

Poles of $G(s)$		Zeros of $G(s)$	
p_1	$= -100$	z_1	$= -183.5$
$p_{2,3}$	$= -10 \pm 50j$	$z_{2,3}$	$= -5.06 \pm 36.95j$
$p_{4,5}$	$= -10 \pm 16j$	$z_{4,5}$	$= -8.84 \pm 12.16j$
$p_{6,7}$	$= -10 \pm 15j$	$z_{6,7}$	$= -9.97 \pm 0.573j$
$p_{8,9}$	$= -10 \pm 10j$	$z_{8,9}$	$= -10.29 \pm 15.53j$
$p_{10,11}$	$= -10 \pm j$	$z_{10,11}$	$= -10.8 \pm 6.315j$
p_{12}	$= -10$		

To study the modal dominance of the system, we evaluated the dominance indices R_i of Litz computed with (5), the modal costs $|V_i|$ of Skelton and Yousuff computed with (5) and the HSV $\sigma_1^{(i)}$ used as new dominance measures (also the Hankel norms of the corresponding subsystems G_i):

Poles of $G(s)$		R_i	$ V_i $	$\sigma_1^{(i)}$
p_1	$= -100$	1.0	120.838	0.5000
$p_{2,3}$	$= -10 \pm 50j$	1.0	156.811	2.5495
$p_{4,5}$	$= -10 \pm 16j$	1.0	68.204	0.9434
$p_{6,7}$	$= -10 \pm 15j$	1.0	66.171	0.9014
$p_{8,9}$	$= -10 \pm 10j$	1.0	55.043	0.7071
$p_{10,11}$	$= -10 \pm j$	1.0	40.901	0.5025
p_{12}	$= -10$	1.0	40.717	0.5000

By inspecting these values we observe that all 12 poles of $G(s)$ have identical dominance indices $R_i = 1$ and for all eigenvalues both the modal costs and HSV have the same order of magnitude. It is clear that at the level of 1×1 blocks, all these dominance measures provide no useful information for an order reduction based on the modal approach. In what follows we shall illustrate how it is still possible to overcome this difficulty and to obtain a substantial reduction of order with the help of the proposed enhanced modal approach.

We construct first a state-space realization for $G(s)$ such that the resulting A is block-diagonal with four diagonal blocks A_1, \dots, A_4 such that

$$\begin{aligned}\Lambda(A_1) &= \{p_1\} \\ \Lambda(A_2) &= \{p_2, p_3\} \\ \Lambda(A_3) &= \{p_4, p_5, p_6, p_7, p_8, p_9\} \\ \Lambda(A_4) &= \{p_{10}, p_{11}, p_{12}\}\end{aligned}$$

This construction was done automatically by using the minimal realization and the block-diagonalization functions available in the MATLAB toolbox HTOOLS [22]. The HSV corresponding to the four TFMs G_1, \dots, G_4 are:

G_1	G_2	G_3	G_4
$\sigma_1^{(1)} = 0.5$	$\sigma_1^{(2)} = 2.5$ $\sigma_2^{(2)} = 2.5$	$\sigma_1^{(3)} = 2.6832$ $\sigma_2^{(3)} = 0.7882$ $\sigma_3^{(3)} = 0.0228$ $\sigma_4^{(3)} = 0.019$ $\sigma_5^{(3)} = 4.92 \cdot 10^{-5}$ $\sigma_6^{(3)} = 2.13 \cdot 10^{-5}$	$\sigma_1^{(4)} = 1.4827$ $\sigma_2^{(4)} = 4.95 \cdot 10^{-6}$ $\sigma_3^{(4)} = 2.1 \cdot 10^{-5}$

We see that four eigenvalues of A_3 and two eigenvalues of A_4 are clearly non-dominant. Thus, with a tolerance value set to say $\varepsilon = 0.05$, we determined a sixth order reduced model containing the dominant modes of the system. The model reductions performed on G_3 and G_4 have been done by using the square-root B&T method of [17]. For referencing purposes we give the numerical values of both the resulting state-space model

$$\left[\begin{array}{c|c} A_r & B_r \\ \hline C_r & D_r \end{array} \right] = \left[\begin{array}{cccccc|c} -100 & 0 & 0 & 0 & 0 & 0 & -10 \\ 0 & -0.0005 & 51.0936 & 0 & 0 & 0 & -0.0517 \\ 0 & -50.8868 & -19.9995 & 0 & 0 & 0 & 9.9999 \\ 0 & 0 & 0 & -17.0175 & 15.9998 & 0 & 9.5563 \\ 0 & 0 & 0 & -15.9998 & -6.3853 & 0 & 3.1727 \\ 0 & 0 & 0 & 0 & 0 & -10.1335 & 5.4818 \\ \hline 10 & -0.0517 & 9.9999 & 9.5563 & -3.1727 & 5.4818 & 0 \end{array} \right]$$

as well as the corresponding transfer function

$$G_r(s) = -\frac{100}{s+100} + \frac{100s}{s^2+20s+2600} + \frac{81.257s+1382}{s^2+23.4s+364.6} - \frac{10.066}{s+6.3853}$$

We can now evaluate, along the lines of the analysis methodology outlined in [6], some performances of the resulting reduced model in order to compare them with those of the original model. The poles and zeros of the resulting $G_r(s)$ are:

Poles of $G_r(s)$	Zeros of $G_r(s)$
$p_1 = -100$	$z_1 = -184.34$
$p_{2,3} = -10 \pm 50j$	$z_{2,3} = -5.08 \pm 37.04j$
$p_{4,5} = -11.701 \pm 15.091j$	$z_{4,5} = -11.75 \pm 7.40j$
$p_6 = -10.133$	

It is apparent that the model reduction preserved the minimal phase behaviour of the original system as well as the relative poles-zeros excess. The step responses and the Bode plots of the original system and of the reduced system are practically indistinguishable. The error between the DC-gains is

$$|G(0) - G_r(0)| = 0.0028,$$

being less than the error resulting for the same order B&T approximation.

We computed also the following relative frequency response errors for three different norms:

$$\frac{\|G - G_r\|_\infty}{\|G\|_\infty} = 0.043, \quad \frac{\|G - G_r\|_H}{\|G\|_H} = 0.0402, \quad \frac{\|G - G_r\|_2}{\|G\|_2} = 0.1304$$

All our analysis indicates that the computed sixth order reduced model is a very good approximation of the original system.

Note. The sixth order approximation considered in this section was intended only to illustrate the main aspects of the proposed enhanced modal approach. Satisfactory fourth order approximations for the same example are described in [10] and [5]. With the above partition of the state matrix, the computation of a fourth order approximation is possible with our method too, but with a sensibly larger tolerance ($\varepsilon = 0.6$). However, by merging the last two blocks to form a larger ninth order block, a more satisfactory fourth order approximation can be computed using a lower tolerance.

7 Conclusions

A model reduction procedure based on an enhanced modal dominance analysis technique has been proposed. The proposed procedure fulfills the basic requirements (generality, numerical reliability, enhanced accuracy) for a satisfactory numerical algorithm and thus can serve as basis for robust software implementation. The new procedure extends the range of applicability of the modal approach to the reduction of arbitrary continuous- or discrete-time systems. In the same time, it can be seen as enlarging also the applicability of many powerful model reduction methods to very large order systems. Implementations of the proposed modal approach in conjunction with two balancing-free square-roots method are available in the last version of the HTOOLS package [22].

References

- [1] Bavely, C., and Stewart, G. W. An algorithm for computing reducing subspaces by block diagonalization. *SIAM J. Control Optim.* 16 (1979), pp. 359–367.

- [2] Chidambara, M. R. Further remarks on simplifying linear dynamic systems. *IEEE Trans. Autom. Control* 12 (1967), pp. 213–214.
- [3] Chu, P. Y., Wie, B., Gretz, B., and Plescia, C. Approach to large space structure control system design using traditional tools. *AIAA Journal of Guidance, Control, and Dynamics* 13 (1990), pp. 874–880.
- [4] Davison, E. J. A method for simplifying linear dynamic systems. *IEEE Trans. Autom. Control* 11 (1966), pp. 93–101.
- [5] Fasol, K. H., and Gehre, G. Order reduction, model approximation and controller design. *Systems Analysis Modelling and Simulation* 8 (1991), pp. 485–505.
- [6] Fasol, K. H., and Varga, A. Zur Ordnungsreduktion linearer systeme. *Automatisierungstechnik* 41 (1993), pp. 6–18.
- [7] Fossard, A. On a method for simplifying linear dynamic systems. *IEEE Trans. Autom. Control* 15 (1970), pp. 261–262.
- [8] Glover, K. All optimal Hankel-norm approximations of linear multivariable systems and their L^∞ -error bounds. *Int. J. Control* 39 (1974), pp. 1115–1193.
- [9] Hammarling, S. J. Numerical solution of the stable, non-negative definite Lyapunov equation. *IMA J. Numer. Anal.* 2 (1982), pp. 303–323.
- [10] Kiendl, H., and Post, K. Invariante Ordnungsreduktion mittels transparenter Parametrierung. *Regelungstechnik* 36 (1988), pp. 92–101.
- [11] Litz, L. Ordnugsreduktion linearer Zustandsraummodelle durch Beibehaltung der dominanten Eigenbewegungen. *Regelungstechnik* 27 (1979), pp. 80–86.
- [12] Litz, L. Praktische Ergebnisse mit einem neuen modalen Verfahren zur Ordnungsreduktion. *Regelungstechnik* 27 (1979), pp. 273–280.
- [13] Marschall, S. A. An approximate method for reducing the order of a linear system. *Contr. Eng.* 10 (1966), pp. 642–648.
- [14] Moore, B. C. Principal component analysis in linear system: controllability, observability and model reduction. *IEEE Trans. Autom. Control* 26 (1981), pp. 17–32.
- [15] Skelton, R. E., Hughes, P. C., and Hablani, H. B. Order reduction for models of space structures using modal cost analysis. *AIAA Journal of Guidance, Control, and Dynamics* 5 (1982), 351–357.
- [16] Skelton, R. E., and Yousuff, A. Component cost analysis of large systems. *Int. J. Control* 37 (1983), pp. 285–304.

- [17] Tombs, M. S., and Postlethwaite, I. Truncated balanced realization of a stable non-minimal state-space system. *Int. J. Control* 46 (1987), pp. 1319–1330.
- [18] Van Dooren, P. The generalized eigenstructure problem in linear systems theory. *IEEE Trans. Autom. Control* 26 (1981), pp. 111–129.
- [19] Varga, A. Efficient minimal realization procedure based on balancing. El Moudni A., Borne P., and Tzafestas S. G. (Eds.): *Prepr. of IMACS Symp. on Modelling and Control of Technological Systems* vol. 2, pp. 42–47, 1991.
- [20] Varga, A. Minimal realization procedures based on balancing and related techniques. Pichler F. and Diaz R. M. (Eds.): *Computer Aided Systems Theory - EUROCAST'91*, vol. 585 of *Lect. Notes Comp. Scie.* Springer Verlag, Berlin, pp. 733–761, 1992.
- [21] Varga, A. Numerical methods and software tools for model reduction. Troch I. and Breitenecker F. (Eds.): *Proc. of 1st MATHMOD Conf., Viena* vol. 2, pp. 226–230, 1994.
- [22] Varga, A., and Ionescu, V. HTOOLS - A Toolbox for solving H_∞ and H_2 synthesis problems. In *Proc. of IFAC/IMACS Symp. on Computer Aided Design of Control Systems, Swansea, UK* pp. 508–511, 1991.