Lecture notes

Grundlagen der Aeroakustik
(Basics of Aeroacoustics)

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Introduction

Among the various disciplines of classical mechanics, ”Aeroacoustics” is a comparatively young subject area. This is mainly because of the fact that it took until 1952, when Lighthill formulated his famous aeroacoustic wave equation and thus ”founded” the field of modern aeroacoustics. With the help of the source terms in Lighthills equation it became possible to physically understand for the first time the origin of sound produced by free turbulence. The derivation of this very equation initiated many developments extremely important for the understanding of aeroacoustic sound generation in general. These notes are an attempt to provide a very brief overview about these essential aspects of the theoretical foundations of aeroacoustics, at least from the author’s point of view. The aim is to convey to the reader the main ideas needed in order to understand the pertinent literature. The above selection of ”key-concepts” of aeroacoustics is necessarily non-unique, and many certainly interesting aspects had to be abandoned in order to fit the lecture into the given restricted time frame of one semester.

Synopsis

Some examples of ”noise generated aerodynamically” in section 1 are to make the reader aware of the variety of technical problems associated with aeroacoustics. Section 2 introduces to the subject area ”acoustics” in general, including various important definitions. Here the acoustic wave equation, describing the sound propagation in non-moving media is derived as a special case of the equations governing the dynamics of small perturbations in a flow field. Some of the very fundamental solutions of the wave equation are derived and discussed insofar as they illustrate the physics of sound generation and radiation as well as the motivation for the solution approaches in general cases. Then the general solution method of Green’s functions is introduced including some essential features of the theory of generalized functions, which are extremely useful for the mathematical handling of complex situations such as the description of the sound generation by moving bodies of arbitrary (and variable) shape. This section also contains the description of surface sound interaction. Moreover the Kirchhoff integral as basis for wave extrapolation is introduced. Next, the multipole-expansion of sound sources is briefly looked at. Then, the concept of acoustic nearfield, farfield and compactness along with the physical implications is introduced. Section 3 is devoted to describing sound waves in moving media and the motion of sound sources. The wave equation is generalized to account for various types of flow fields, such as potential flow and parallel shear flow. The sound intensity is generalized for sound propagating in a flow field. Some essential consequences on the propagation and radiation are discussed such as the effect of flow on the propagation of sound waves in ducts, convective amplification, sound refraction in shearlayers/temperature layers and Doppler-frequency shift. In Section 4 the various acoustic wave equations due to Lighthill, Möhring, Lilley and Ffowcs Williams and Hawkings are derived and their source terms are discussed in view of a physical interpretation. Finally, secion 5 concludes the course with a discussion of some technical applications.

List of Symbols

f	–	scalar function
$\boldsymbol{f}$	–	vector function (bold face)
$\boldsymbol{x}$	–	spatial co-ordinates (usually position vector of observer)
$\boldsymbol{\xi}$	–	spatial co-ordinates (usually position vector of source)
t	–	time (usually observer time)
τ	–	time (usually source time or retarded time)
$\boldsymbol{\nabla}\cdot$	–	”divergence of”
$\boldsymbol{\nabla}$	–	”gradient of”
$\boldsymbol{\nabla}\times$	–	”curl of”
$\cdot$	–	simple contraction
$:$	–	double contraction
Δ	–	Laplace operator $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$
$\frac{D}{Dt}$	–	total (or material) derivative w.r.t. time (following a fluid particle)
δ	–	total variation of a thermodynamic variable

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1 Aeroacoustic Phenomena

Aeroacoustics deals with the sound generated aerodynamically. Certainly, there are many coupled problems, in which unsteady fluid loading on structures causes them to vibrate and radiate sound. In a way, one may think of this also as ”sound generated aerodynamically”, because the loading then is of aerodynamic origin. But what we really mean, is the noise generated in the unsteadily moving fluid itself, and the influence of (non-vibrating, stiff) bodies on this very process. There is a variety of applied problems where aeroacoustic phenomena are the major contributors to noise, i.e. aircraft engine noise, propeller noise, rotor noise, noise of fans (ducted and unducted) in whatsoever application, cavity noise, tones of wires and cylindrically shaped bodies in transverse wind, airframe noise at aircraft, high speed trains and cars, noise in valves and nozzles, etc. In the following we will try to explain the nature of all these phenomena theoretically employing the well established theory of aeroacoustics, which was initiated by Lighthill in 1952 [2].

2 Basic concepts of acoustics

2.1 Definitions

2.1.1 General terms

The field of acoustics contains numerous parts of which important ones are listed below:

- physical acoustics: Dynamics of continua
  - building acoustics (vibrational excitation of noise from the floor, sound transmission through walls,...)
  - room acoustics (design of concert halls, tone studios, ...)
  - aero-/flow acoustics (flow induced noise)
- electro acoustics: generation, amplification, measurement
- musical acoustics: music, instruments
- physiological acoustics: functioning of the ear
- psycho acoustics: perception investigations with respect to different sound events
- technical acoustics: generic term for all acoustic issues, which are connected to technical products

2.1.2 Acoustic quantities

The perception of sound is the response to a physical stimulus to the ear. This stimulus is the unsteady *sound pressure*

$$p'(t) := p(t) - p^0 \tag{1}$$

in Pascal. The steady part p^0 of the pressure $p(t)$ is the temporal average

$$p^0 = \overline{p} := \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} p(t)dt \tag{2}$$

and is not perceived auditively. More strictly $p'(t)$ should be termed ”fluctuating sound pressure” (german: ”Schallwechseldruck”). In practice the time average has to be taken using a finite averaging period T.

$$p^0 \approx \overline{p}^{T} := \frac{1}{T} \int_0^T p(t)W(t,T)dt \tag{3}$$

with an appropriate weighting (”window”) function $W(t,T)$. Generally, if T is chosen large enough then most of p^0 is composed of ”infra sonic” components, i.e. components with frequencies of the order of 16Hz or lower, which are inaudible to the human ear. If T is chosen to be e.g. 0.5 seconds, then all frequency components over 16Hz originally contained in $p(t)$ are suppressed by more than 96% in p^0 for the simplest choice of $W = 1$, see appendix A. This in turn means that the largest deviation in p' due to the finiteness of T is less than 4% for frequency components above 16Hz. For $W = 1 + \cos(2\pi t/T)$ this suppression may for instance be enhanced to more than 99.9%.

An appropriate measure for the strength of an acoustic signal is the *root mean square or ”rms” value* (german: ”Effektivwert”) $\tilde{p}$ of the sound pressure

$$\tilde{p} = \sqrt{\overline{(p')^2}} \tag{4}$$

For instance, the rms-value of a sinusoidal sound pressure signal $p'(t) = \hat{p}\cos(\omega t)$ due to eqn.(4) is $\tilde{p} = \sqrt{1/2}\,\hat{p}$, thus the rms-value is indeed a measure for the strength (amplitude) of a signal. The ear is capable to resolve a tremendously wide range of orders of magnitude of the sound pressure:

$\tilde{p}_{\min} \approx 10^{-5}$ Pa - is the smallest perceivable pressure amplitude and called *threshold of hearing*, which is frequency dependent[1] (german: ”Hörschwelle”)

$\tilde{p}_{\max} \approx 10^{2}$ Pa - threshold of pain (german: ”Schmerzgrenze”)

Because of the large range of rms-values which are resolved by the ear one has introduced a logarithmic scale for the most important acoustic quantities. The *sound pressure level* L_p or SPL (german: ”Schalldruckpegel”) is defined as

$$L_p := 10\lg\left(\frac{\tilde{p}}{p_\mathrm{ref}}\right)^2 \mathrm{dB} \quad = \quad 20\lg\left(\frac{\tilde{p}}{p_\mathrm{ref}}\right)\mathrm{dB}, \tag{5}$$

$$p_\mathrm{ref} \quad := \quad 2\cdot 10^{-5}\mathrm{Pa}, \tag{6}$$

[1]compare this value with the sound pressure corresponding to the Brownian motion of molecules in air, being $\approx 0.5\cdot 10^{-5}$ Pa, meaning that the threshold of hearing is at a physically reasonable lower limit

where the international reference pressure p_{ref} corresponds to about the threshold of hearing for a sinusoidal signal at roughly 2 kHz. Although a level is dimensionless by definition, the supplement "dB" (decibel after Alexander Graham Bell) indicates that it is a logarithmic measure. It is essential to get familiar with the decibel-scale. For instance a doubling of the rms-value corresponds to an increase of the sound pressure level by $\Delta L_p = 6$ dB, because

$$\Delta L_p = 20\Big[\lg\Big(2\frac{\tilde{p}}{p_{\text{ref}}}\Big) - \lg\Big(\frac{\tilde{p}}{p_{\text{ref}}}\Big)\Big]\ \text{dB} = 20\lg 2\text{dB} = 6.02...\ \text{dB}.$$

An acoustically induced change in pressure is always accompanied by a local motion of the medium. The fluid velocity $\boldsymbol{v}'$ of this motion is called *acoustic particle velocity* (german: "Schallschnelle"). It is entirely different from the "speed of sound" (see below). In order to illustrate this difference one may consider a sound field, produced by an hard sphere, oscillating in the horizonal direction. The right part of figure *1* shows two moments in time of this oscillation along with the corresponding sound field. The latter is visualized by small spheres, which represent Lagrangian fluid elements. While on the one hand these fluid elements oscillate with a small amplitude about a fixed position the fluid compressions and dilatations, represented by the closeness of the elements move at a much higher speed and strictly radially away from the sphere (yellow arrows).

While the sound pressure or the acoustic particle velocity represent the acoustic signal, the speed of sound is the propagation speed of the signal through the medium. In which way the sound pressure and the acoustic particle velocity are linked to one another is discussed in a later section 2.4.2.3. For the moment it suffices to recognize that such motion necessarily exists. The *acoustic*

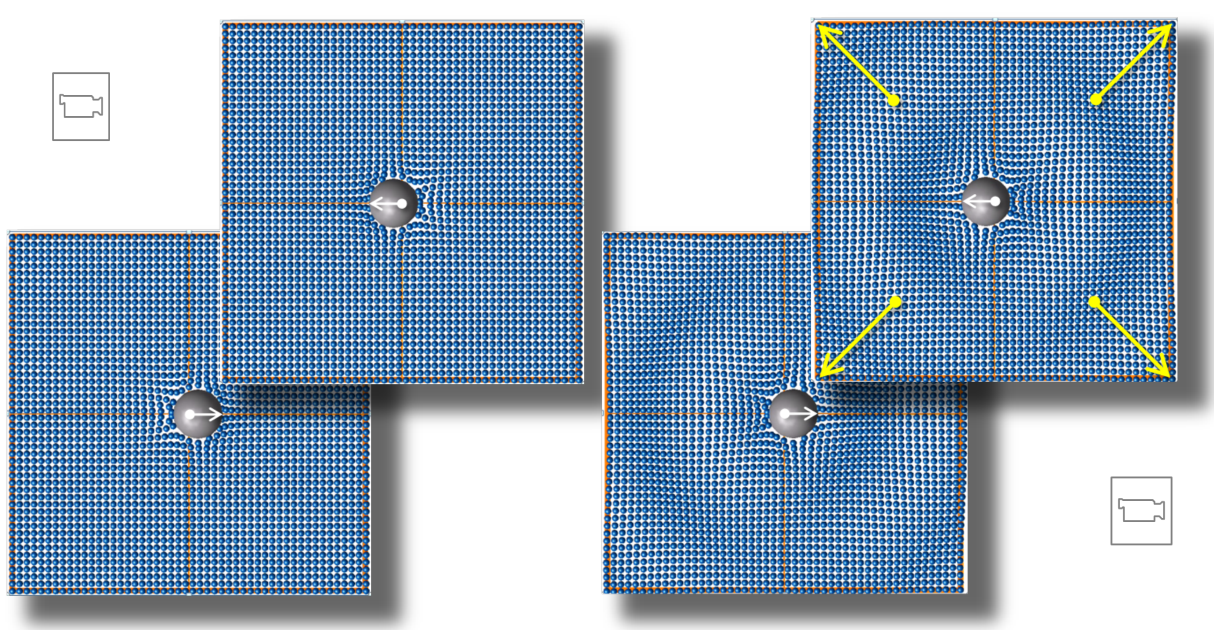

Figure 1: Kinematics of fluid elements induced in the neighborghood of a sphere, oscillating horizontally about it's center (motion exaggerated). Left: incompressible fluid, right: compressible fluid.

particle velocity level (german: ”Schallschnellepegel”) is defined as:

$$L_v := 20 \lg \left(\frac{|\tilde{\boldsymbol{v}}|}{v_{\text{ref}}} \right) \text{dB}, \tag{7}$$

$$v_{\text{ref}} := 5 \cdot 10^{-8} \text{m/s} \tag{8}$$

The *sound intensity* $\boldsymbol{I}$ (german : ”Schallintensität” or ”Energiestromdichte”) in a stagnant medium is derived from the sound pressure and the acoustic particle velocity like

$$\boldsymbol{I}(\boldsymbol{x}) := \overline{p' \boldsymbol{v}'} \tag{9}$$

The corresponding *sound intensity level* L_I (german: ”Schallintensitätspegel”) is defined by

$$L_I := 10 \lg \left(\frac{|\boldsymbol{I}|}{I_{\text{ref}}} \right) \text{dB}, \tag{10}$$

$$I_{\text{ref}} := 10^{-12} \text{W/m}^2 \tag{11}$$

The sound power P (german: ”Schallleistung”) emitted by a source in a quiescent medium is obtained by integration of the sound intensity $\boldsymbol{I}$ over a closed surface A enclosing the very source:

$$P := \oint_A \boldsymbol{I} \cdot \boldsymbol{n} \; dA \tag{12}$$

As can be shown, P describes the sound power of the source even if $\boldsymbol{I}$ is influenced by sound of sources located outside A. As will be shown below, P describes the power of the sound sources located inside A as long as dissipation is negligibly small. Then P is independent of A (as long as the same sources stay enclosed by all surfaces A). The corresponding *sound power level* L_W (german: ”Schallleistungspegel”) is defined as

$$L_W := 10 \lg \left(\frac{P}{P_{\text{ref}}} \right) \text{dB}, \tag{13}$$

$$P_{\text{ref}} := 10^{-12} \text{W} \tag{14}$$

Note: The definition of intensity and power is non-trivial in flowing media (see later).

2.1.3 Characterisation of acoustic events

A clear distinction can be made between different kinds of sound signals. Acoustic signals are therefore categorised into classes:

pure tone (german: ”reiner Ton”)	:	$p(t) = \hat{p}^c \cos(\omega t) + \hat{p}^s \sin(\omega t)$
tone (german: ”Ton”)	:	$p(t) = \sum_{n=0}^{\infty} \hat{p}_n^c \cos(n\omega t) + \hat{p}_n^s \sin(n\omega t)$
complex tone (german: ”Klang”)	:	superposition of tones ($p(t)$ generally non-periodic)
”pure” noise (german: ”Rauschen”)	:	$p(t)$ has a continuous frequency content, the signal is permanent, stochastic
noise (german: ”Geräusch”)	:	superposition of complex tones and ”pure” noise
impulse (german: ”Impuls/Tonimpuls”)	:	short duration sound event (no rms-value definable)
bang (german: ”Knall”)	:	alternating impulse with zero time integral (e.g. sonic boom, N-wave).

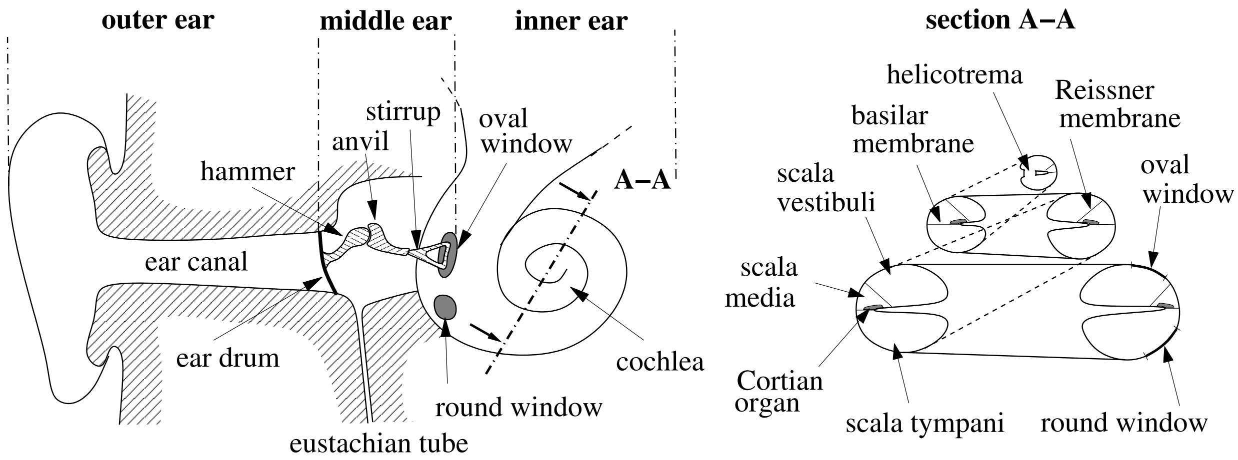

Figure 2: schematic of the ear

2.1.4 Physiology of the ear

The ear is divided into three main parts (see figure *2*):

outer ear (german: "Aussenohr")	:	consists of outer (i.e. visible part of the) ear (german: "Ohrmuschel"), and the ear canal (german: "Gehörgang") with a length of $\approx 2 - 2.5$cm. The ear canal forms a one-sided open tube, i.e. resonances occur for wavelengths $\lambda/4 \approx 2.5$cm corresponding to about $3.4 - 4.2$ kHz. This physiological circumstance is responsible for an enhanced hearing sensitivity in this frequency range.
middle ear (german: "Mittelohr")	:	small bones (german: "Gehörknöchelchen"), transmission of the deformation of the ear drum (german: "Trommelfell") onto the oval window (german: "ovales Fenster").
inner ear (german: "Innenohr")	:	cochlea (german: "Schnecke"), contains 3 coiled canals also "scala".

Incident sound waves are transmitted through the ear canal and cause the ear drum to move. This movement is transmitted through three small bones (hammer, anvil and stirrup). The hammer (german: "Hammer") is connected to the ear drum, while the stirrup (german: "Steigbügel") is attached to the deformable oval window inside the cochlea, which is made up of very stiff bone. The anvil (german: "Amboss") connects both hammer and stirrup to form a mechanical transducer adapting the impedance of the air with the considerably higher impedance of the lymphatic fluid inside the cochlea.

The coiled cochlea contains three parallel canals (*scala*), namely *scala vestibuli*, *scala media* and *scala tympani*. The *scala vestibuli* is connected to the oval window, while the *scala tympani* is connected to the (deformable) round window. These two parallel coiled canals are separated by a stiff boned thin wall with a slit alongside the canals. The slit is closed by the flexible *Basilar membrane*. The flexible Reissner membrane forms the third separate canal, the scala media. Only at the end of the canals an opening, the so called *helicotrema*, connects them.

For very low frequency movements of the oval window, the fluid in the cochlea is displaced along the scala vestibuli, through the helicotrema and further along the scala tympani, at whose end it deforms the round window correspondingly. When higher frequency excitations deform the oval window, then the inertia of the fluid in the cochlea causes it to take a ”short cut” from oval window to round window. This is achieved by locally deforming the Basilar membrane (see figure 3). As can be seen in the sketch the location where this very deformation takes place is depending on the frequency. In this way a frequency-position transformation takes place along the canal of the cochlea. The deformation of the Basilar membrane is sensed in the so called Cortian organ, which is sitting on the membrane. The location of the membrane's deformation determines the frequency of the acoustic signal and thus whether the tone is perceived as a high or low pitch.

The sensation of hearing is limited to a frequency range of

$$16\text{Hz} < f < 20000 Hz \tag{15}$$

In order to sense the (horizontal) direction out of which sound impacts the observer two ears are required. The mechanism by which this capability of directional hearing (german: ”Richtungswahrnehmung”) is achieved is frequency dependent.

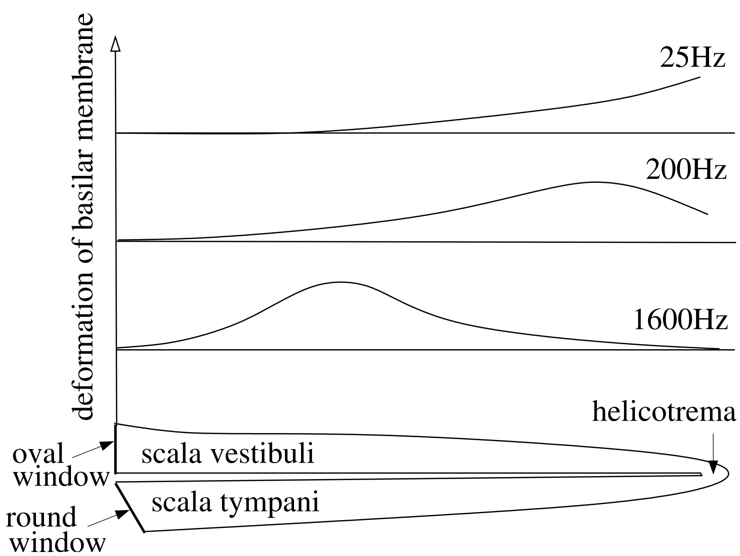

Figure 3: Deformation of the Basilar membrane (unwound cochlea)

$f > 1.5$kHz $\rightarrow$ shielding effect of the head. As an obstacle the head produces a "shadow" region on the side opposite to the direction of incidence. This mechanism requires the wavelength of the sound to be sufficiently small. Considerable shielding effects occur when the wavelength is smaller than the characteristic dimensions of the shielding body. This explains the lower frequency limit of $f = 1.5$kHz, which corresponds to a wavelength of about 0.23m, which in turn corresponds roughly to the diameter of the head.

$f < 1.5$kHz $\rightarrow$ temporal delay of signal at the two ears. Depending on the direction of incidence the sound travels paths of different length to either ear. This leads to a phase difference of the signal at the two ears. The same principle is used for the localization of sound sources with the help of so called microphone arrays. Such arrays are composed of quite more than two microphones ("ears"). Typically some two hundred microphones or so are used leading to a high resolution of localization (see lecture "Methoden der Aeroakustik" for reference).

It is noted that by construction the ability to detect the direction of incidence of a sound wave emitted from a source located in the (vertical) symmetry plane of the head is weak. The non-circular shape of the outer ear somewhat helps in combination with indirect effects as e.g. reflections at shoulders, which somewhat alter the incident signal depending on whether it arrives from the front or the back.

2.1.5 Acoustic quantities adapted to hearing

2.1.5.1 Loudness and Loudness level The sensitivity of the human ear is strongly dependent on the frequency of the perceived sound. Therefore two pure tones with the same sound pressure level but with different frequencies will be perceived as differently loud. The so called *loudness level* (german: "Lautstärkepegel") is defined to account for this observation. The definition of the loudness is restricted to pure tones with frontal incidence.

The loudness level L_N of a pure tone of a fixed frequency $f = \omega/(2\pi)$ and given sound pressure level $L_p(f)$ is defined as the sound pressure level of the pure 1000Hz tone, which is perceived "equally loud".

Diagram *4* shows lines of constant loudness level, so called *equal loudness level contours, ELLC*. The ELLC are defined in ISO226. It was determined based on data of a large group of test persons. The units of loudness level are called *phons*. By definition the phon-values are equal to the sound pressure levels in dB for $f_{\mathrm{ref}} = 1000$Hz. For frequencies lower than 1000Hz the diagram shows that the L_p-value is higher than the L_N-value: $L_p(f < f_{\mathrm{ref}}) > L_N(f)$. This is a consequence of the decrease in hearing sensitivity in the low frequency range. For frequencies larger than 1000Hz up to about 6kHz the L_p-value is smaller than the L_N-value with a minimum at about 4000Hz because due to the physiology of the ear, in this range the hearing sensitivity is particularly good. Beyond about 6kHz the values of the sound pressure level become larger than the values of the loudness again as a consequence of the reduced hearing sensitivity in the larger frequency range. At roughly 10-12kHz a second distinct minimum in the ELLC occurs, which may again be explained with an increased hearing sensitivity due to the length of the ear canal

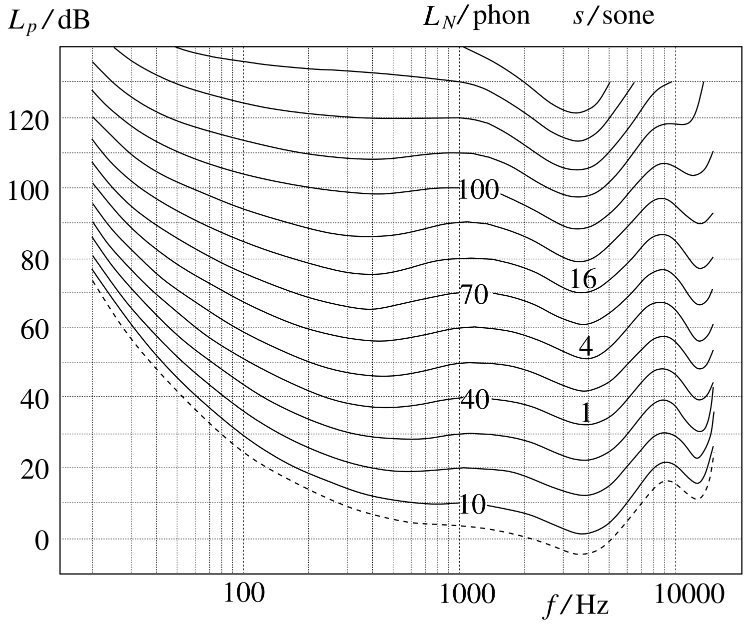

Figure 4: Loudness level contours L_N =const. Right numbering: sone values.

equalling 3/4 of the respective sound wavelength.

The hearing sensitivity is not only frequency dependent; there is a strong nonlinear dependence of the perception of intensity differences. A doubling of the intensity of a tone is not perceived "double as loud". Typically an increase of 10 phons (or roughly 10dB), corresponding to a tenfold sound intensity (or mean square pressure), is perceived only "double as loud". In order to account for this characteristic of hearing the so called *loudness* s (german: "Lautheit") was introduced. The loudness is defined as

$$s := 2^{(L_N-40)/10}, \quad [s] = \text{sone} \tag{16}$$

and is a measure for the intensity perception. From the above defintion the loudness ratio of two sounds with a loudness level difference of ΔL_N is $s_2/s_1 = 2^{(L_{N2}-40)/10-(L_{N1}-40)/10} = 2^{(L_{N2}-L_{N1})/10} = 2^{\Delta L_N/10}$. For instance an increase of a signal by 10 phons in loudness level or (roughly 10dB in sound intensity level) corresponds to a doubling of the loudness; an increase of 20 phons corresponds to a fourfold loudness value. Of course, given a loudness value, the loudness level may be computed from eq.(16):

$$L_N = 40 + 33.2 \lg s \quad . \tag{17}$$

2.1.5.2 Noise weighting From the preceding notes it is obvious that on the one hand the physical sound pressure or sound intensity levels are somewhat inappropriate measures for the characterization of some perceived sound. On the other hand the complex dependence of the hearing sensitivity on a large number of influences makes it almost impossible to condense all of this into one single value. Nevertheless in practice it is necessary to use simple measures (even sometimes at the expense of considerable simplification). This is where the so called *noise weighting* (german: "Schallbewertung") comes into play. In order to take the hearing sensitivity into account in a simple way the sound pressure level L_p is weighted in a frequency dependent way. For the

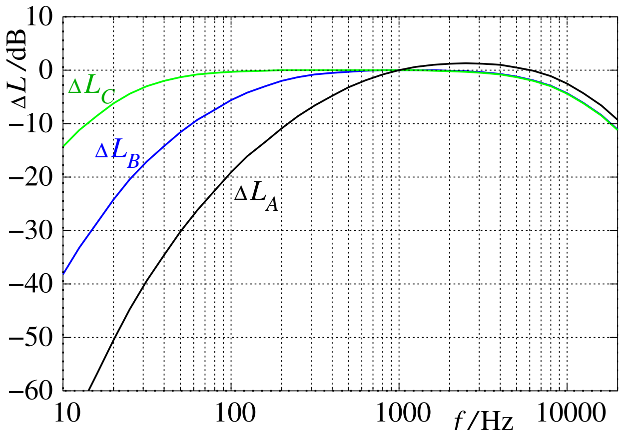

Figure 5: Noise weighting curves, DIN-IEC 651.

moment let us assume that the underlying signal is a pure tone of frequency f, although it will turn out later, that we may relax this restriction to general (permanent) sounds when introducing the Fourier analysis. Very high frequency and very low frequency signals are downweighted, while mid-frequency signals (i.e. those in the highest hearing sensitivity) are upweighted. Internationally at least 4 noise weightings were agreed (called A-, B-, C- and D-weighting), of which the A-weighting has turned out to be most widely applicable.

$$\begin{aligned} L_p &\longrightarrow L_{pA}, L_{pB}, L_{pC}, L_{pD} \\ dB &\longrightarrow dB(A), dB(B), dB(C), dB(D) \end{aligned}$$

where

$$L_{pA} = L_p + \Delta L_A \tag{18}$$

The definitions are analogous for the B-,C-, and D-weightings. The weighting $\Delta L_A = \Delta L_A(f)$ was designed for low to moderately intense SPL (below 55dB). It roughly corresponds to the negative loudness level curve $L_N = 40$ phons. It may be parametrized approximately with the polynomial $\Delta L_A \approx -145.528 + 98.262 \lg f - 19.509(\lg f)^2 + 0.975(\lg f)^3$. The B-(or C-)weighting corresponds roughly to the negative loudness level curve L_N =70 (or 100) phons. The B- and C-weighting correspondingly were defined for situations when the noise is of very high intensity (B for SPL=55-85dB, C for SPL $>$ 85dB). The D-weighting was designed especially for aviation noise. However, it is not widely used.

2.1.5.3 Narrow-, third octave and octave band analysis So far we have defined adapted acoustic quantities only for pure tones. In most practical situations the signal will be quite more general though. Usually one has to deal with general fluctuating signals, so called random processes.

Some statistics of random processes

An example for some random process (e.g. an acoustic flyover or pass-by signal) would be like depicted in figure 6 for a quantity $h(t)$, representing either the pressure p or the velocity $\boldsymbol{v}$. It is quite obvious that it may not be appropriate for any function to use the definition eqn(2) to define an average or a mean value. For a random function h one defines a statistical or so called

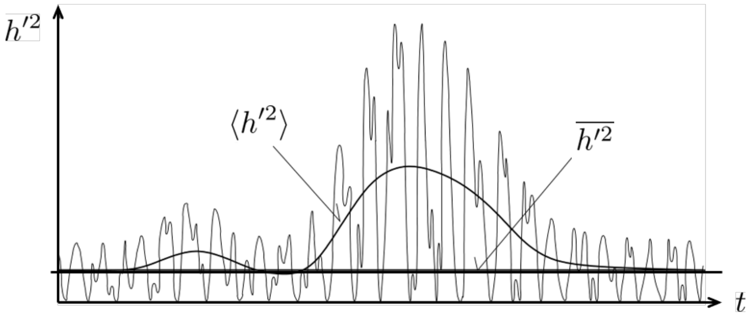

Figure 6: Example for a randomly varying function.

ensemble average

$$\langle h\rangle(t) := \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N} h_n(t), \tag{19}$$

where $h_n(t)$ is called the n'th realisation (practically the n'th measurement) of the process $h(t)$. Analogously to (2) for $h = p$ we define the fluctuating pressure $p' = p - \langle p\rangle$ and correspondingly $\langle p'\rangle = 0$, while $\langle p'^2\rangle \neq 0$, or the acoustic particle velocity $\boldsymbol{v}' = \boldsymbol{v} - \langle \boldsymbol{v}\rangle$ with $\langle \boldsymbol{v}'\rangle = 0$, while $\langle \boldsymbol{v}'^2\rangle \neq 0$ respectively.

A statistical measure for a random variable h', whose utmost importance will become obvious below, is the *auto correlation* $C_{hh}(t,\tau)$ (german: "Autokorrelation") of $h'(t)$ being the statistical average of the product with itself, but shifted by an arbitrary time shift variable τ:

$$C_{hh} := \langle h'(t)h'(t+\tau)\rangle \tag{20}$$

For instance, in the context of this lecture the most important physical quantity to characterize will be the pressure $h' = p'$. In analogy to the autocorrelation the correlation between two different random variables h' and g' may be evaluated and is called *cross correlation* $C_{hg} = C_{gh}$ (german: "Kreuzkorrelation") of $g'(t)$ and $h'(t)$:

$$C_{hg} := \langle h'(t)g'(t+\tau)\rangle \tag{21}$$

We already came across the sound intensity, representing the product of two different acoustic variables namely pressure and velocity, in which the cross correlation comes into play in case $h' = p'$ and $g' = \boldsymbol{v}'$ exhibit a random behavior.

Many physical processes are random processes but their statistical behavior exhibits certain characteristic features. Processes, which do not change their statistics with time are quite important in applications. Such processes are called *statistically stationary* (german: "statistisch stationär") and for the variables p' and $\boldsymbol{v}'$ of interest are characterized by

$$C_{pp} = \langle p'(t)p'(t+\tau)\rangle = C_{pp}(\tau) \neq C_{pp}(t) \text{ or } \boldsymbol{C}_{pv} = \langle p'(t)\boldsymbol{v}'(t+\tau)\rangle = \boldsymbol{C}_{pv}(\tau) \neq \boldsymbol{C}_{pv}(t) \tag{22}$$

An extremely important example for statistically stationary processes in fluid mechanics and aerodynamic noise are turbulent fluctuations in (non-transient) turbulent flows.

For "reasonable" random functions, for instance the turbulence related pressure fluctuation, the auto correlation $C_{pp}(\tau)$ decays to zero for large τ, i.e. for sufficiently large time differences

$\tau > \Delta T$ the process forgets it's past. In such case one may cut the time history of the signal $p'(t)$ into pieces with $p'_n(t) := p'(t + n\Delta T)$ and consider them as independent realizations of the process. Such statistically stationary processes are called *ergodic process*, in which case the temporal mean as defined in eqn(2) is equal to the ensemble average $\overline{p'^2} = \langle p'^2 \rangle$, or $\overline{p'\boldsymbol{v}'} = \langle p'\boldsymbol{v}' \rangle$ respectively.

In the following we will only deal with random functions describing statistically stationary processes. In view of the preceding paragraphs on the frequency dependence of the hearing sensitivity it should be extremely useful to analyse any random sound pressure signal into its frequency components. The decomposition of a function $h'(t)$ into its frequency components is called *Fourier Analysis* or *Fourier Transform*

$$\hat{h}(\omega) := \int_{-\infty}^{\infty} h'(t) \exp(-i\omega t)\, dt \tag{23}$$

where $f = \omega/2\pi$ is the frequency and $i = \sqrt{-1}$. It must be noted though, that $\hat{h}$ exists as an ordinary function only if $h'(t)$ is square integrable, i.e. $\int_{-\infty}^{\infty} |h'(t)|^2\, dt < \infty$. The inverse transformation back to real space is

$$h'(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(\omega) \exp(i\omega t)\, d\omega \tag{24}$$

which clearly shows that $h'(t)$ is nothing but the summation of all its frequency components $\hat{h}(\omega)$.

Unfortunately the Fourier-decomposition of our random function $p'^2(t)$, or $p'(t)\boldsymbol{v}'(t)$, is not easily possible because it certainly violates the condition to be square integrable. This is the reason why one resorts to the Fourier decomposition of the auto correlation $C_{pp}(\tau)$ for the pressure or the cross correlation $\boldsymbol{C}_{pv}(\tau)$ for the intensity rather than the random function itself.

Spectrum of sound pressure

The question we want to answer is, how to determine the frequency components of the mean square sound pressure $\tilde{p}^2$. For this purpose we take the auto correlation $C_{pp}(\tau) = \langle p'(t)p'(t+\tau) \rangle$. We assume that $C_{pp}(\tau)$ decays to zero sufficiently fast in τ to be square integrable such that we may apply the Fourier decomposition eqn (23) with τ instead of t (why not trying?) and obtain

$$\hat{C}_{pp}(\omega) = \int_{-\infty}^{\infty} C_{pp}(\tau) \exp(-i\omega\tau)\, d\tau = 2 \int_{0}^{\infty} C_{pp} \cos(\omega\tau)\, d\tau, \tag{25}$$

which is called *power spectral density* (german: "Leistungsdichtespektrum")[2]. Note, we may reduce the integration range to positive τ only, since the autocorrelation by definition is an even

[2] Practically it may not be so simple to determine the power spectral density according to its definition eqn(25). For a statistically stationary signal we outline shortly a simple practical way to do so. First we define an auxiliary function $\hat{p}_T(\omega) := \int_{-T/2}^{T/2} p'(t) \exp(-i\omega t)\, dt$, Next, we determine the time average over p'^2_T as $\overline{p'^2}^T = T^{-1} \int_{-T/2}^{T/2} p'^2\, dt$. Now, $p'(-T/2 \leq t \leq T/2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}_T(\omega) \exp(i\omega t)\, d\omega$ which may be used in the above expression to give $\overline{p'^2}^T = (2\pi T)^{-1} \int_{-\infty}^{\infty} \hat{p}_T(\omega) \int_{-T/2}^{T/2} p'(t) \exp(i\omega t)\, dt\, d\omega = (2\pi T)^{-1} \int_{-\infty}^{\infty} \hat{p}_T(\omega)\hat{p}_T(-\omega)\, d\omega$. We may now let $T \to \infty$ and obtain $\overline{p'^2} = \tilde{p}^2 = \int_{-\infty}^{\infty} \lim_{T\to\infty} |\hat{p}_T(\omega)|^2/T\, d\omega/(2\pi)$. By comparison with eqn(26) one may extract that for a statistically stationary signal its power spectral density $\hat{C}_{pp}(\omega) = \lim_{T\to\infty} |\hat{p}_T(\omega)|^2/T$ may be computed from the Fourier transform of the fluctuation pressure $p'(t)$ over finite observation times T in the limit $T \to \infty$.

function of τ. Now it is quite trivial to turn back and formally write down the inverse transformation for $C_{pp}(\tau)$ according to eqn(24). Since $C_{pp}(\tau = 0) = \tilde{p}^2$ the mean square of the sound pressure finally appears as the integral over the signal power components contained in each frequency (represented by the power spectral density):

$$\tilde{p}^2 = C_{pp}(\tau{=}0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{C}_{pp}(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{\infty}\hat{C}_{pp}(\omega)\,d\omega = 2\int_{0}^{\infty}\hat{C}_{pp}(f)\,df \tag{26}$$

As before, we reduced the integration range to positive ω, using the fact that the power spectral density of any autocorrelation is real and an even function of ω, see right hand side of eqn(25). Now, eqn (26) directly shows that for each (infinitesimal) frequency interval df there is a certain frequency contribution to $\tilde{p}$. Once the power spectral density is known each frequency component of the random signal may be considered separately (especially for the frequency weighting, see below 2.1.5.4).

In order to evaluate the frequency contributions to the sound pressure level or the intensity level of the overall signal, one has to integrate over finite frequency intervals instead of infinitesimal ones. Only the integration of the power spectral density over a finite frequency range will yield (i) a quantity with the dimension of a square pressure, needed to form the sound pressure level, and (ii) a finite value representing the contribution of this frequency range to the overall rms-value of our random like acoustic signals. Hence, the integral over the power spectral density eqn(26) is divided into frequency intervals or consecutive *frequency bands* $f_i^l < f < f_i^u$ (with $f_i^u = f_{i+1}^l$ and $f_i^l = f_{i-1}^u$), implying that the so called *energetic sum* $\tilde{p}^2 = \sum_{i=-\infty}^{\infty} \tilde{p}_i^2$ is made up of the summands (see fig.7):

$$\tilde{p}_i^2 := 2\int_{f_i^l}^{f_i^u}\hat{C}_{pp}(f)\,df \Rightarrow L_p^i := 10\lg\left(\tilde{p}_i^2/p_{\mathrm{ref}}^2\right)\mathrm{dB}. \tag{27}$$

On the right hand side so called *band levels* L_p^i, were introduced, which naturally follow from the availability of the bandwise mean square values. The bandwidth of the i^{th} frequency band $\Delta f_i = f_i^u - f_i^l$ is defined by its lower and upper frequency limits f_i^l and f_i^u respectively. The location of the i^{th} band on the frequency axis is characterized by its *center frequency* f_i^c. Actual conventions for the definition of bands are presented below.

Spectra of sound intensity (and sound power)

We obtain the spectra (frequency content) of the sound intensity $\boldsymbol{I} = \overline{p'\boldsymbol{v}'}$ in a very similar way

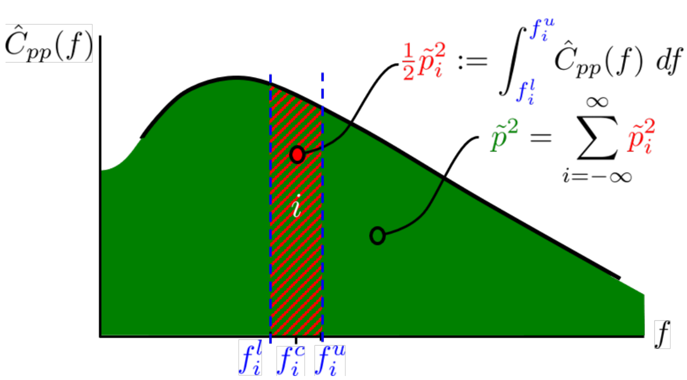

Figure 7: Definition of frequency bands for the determination of the band mean square and band levels.

as for the sound pressure in eqn (26), the only difference being that we cannot assume that its cross spectral density $\hat{C}_{pv}$ is an even function of ω as the power spectral density of the sound pressure was[3]. But we may exploit at least that by definition the cross correlation of p' and $\boldsymbol{v}'$ is real. Therefore, $\hat{C}_{pv}(-\omega)$ ought to be the complex conjugate of $\hat{C}_{pv}(\omega)$ which again enables us to reduce the integration range to only positive frequencies:

$$\boldsymbol{I} = \boldsymbol{C}_{pv}(\tau = 0) = \frac{1}{\pi}\int_0^\infty \Re[\hat{\boldsymbol{C}}_{pv}(\omega)]\, d\omega = 2\int_0^\infty \Re[\hat{\boldsymbol{C}}_{pv}(f)]\, df,$$

where $\Re[.]$ explicitly underlines, that only the real part of $\hat{C}_{pv}$ is to be evaluated. The band levels of the sound intensity and the respective intensity levels are determined analogously to the ones derived for the sound pressure:

$$\boldsymbol{I}_i = (\overline{p'\boldsymbol{v}'})_i = 2\int_{f_i^l}^{f_i^u} \Re[\hat{\boldsymbol{C}}_{pv}(f)]\, df \Rightarrow L_I^i := 10\lg\left(|\boldsymbol{I}_i|/I_{\text{ref}}\right)\text{dB}.$$

Naturally, the sound power P_i for each frequency band f_c^i is determined as in eqn (12) to yield the spectrum of sound power.

Narrow band spectrum
For the characterization of sounds with strong tone components (spectral peaks) so called *narrow band spectra* (german: ”Schmalbandspektrum”) $\tilde{p}_i^2(f_i^c)$ are used. In this case the frequency band $\Delta f_i = \Delta f = const$ and $f_i^c = (f_i^l + f_i^u)/2 = f_0^c + i\Delta f$. Note that the band level of a signal depends on the bandwidth Δf chosen. Since there is no international convention on the choice of some Δf it must always be specified together with a narrow band spectrum.

m-band spectrum
For the characterization of sounds with low intensity tone components and a relatively high pure noise level (e.g. stochastic noise due to turbulence) so called m-band spectra (german: ”m-Bandspektrum”) $\tilde{p}_i^2(f_i^c)$ are used. In this case $f_i^u/f_i^l = 2^m = const$, and the centre frequency $f_i^c = (f_i^l f_i^u)^{1/2} = 2^{mi} f_0^c$. The bandwidth is not constant, but proportional to the center frequency like $\Delta f_i = (2^{m/2} - 2^{-m/2}) f_i^c$. The lower and upper cutoff frequency of the band Δf_i are $f_i^l = 2^{-m/2} f_i^c$ and $f_i^u = 2^{m/2} f_i^c$ respectively. The band level is again defined as $L_{pm}^i := 20\lg(\tilde{p}_i(f_i^c)/p_{\text{ref}})$. For $m = 1/1$ the m-band spectrum is also called *octave band spectrum* (german: ”Oktavspektrum”), because each band spans a whole octave (frequency increase by factor 2). The most often used m-band spectrum is the $m = 1/3$- or *third octave band spectrum* (german: ”Terzspektrum”) though. The respective band level is denoted $L_{p1/3}$. By international standard, the reference centre frequency is $f_0^c = 1$ kHz.

It should be noted, that the practically used centre frequencies of the third octave bands are not really derived from the above formula $f_i^c = 2^{i/3} \cdot 10^3$ Hz. Instead these frequencies are standardized in EN ISO 266 and actually computed like $f_i^c = 10^{i/10} \cdot 10^3$ Hz. Moreover, these centre frequencies are referred to with ”nominal” frequency (german: ”Normfrequenz”) values according to table *1*.

[3] In section 2.4.3, eqn (83) we will see that for an observer located far from the source of the sound field (farfield) $\boldsymbol{I} \propto \tilde{p}^2$, so that in the farfield the cross spectral density of intensity and the power spectral density of pressure become proportional to another.

nominal frequency	exact (ISO 266) $10^{i/10} \cdot 10^3$	third octave $2^{i/3} \cdot 10^3$
$\vdots$	$\vdots$	$\vdots$
400	398.1	396.9
500	501.1	500.0
630	631.0	630.0
800	794.3	793.7
1000	1000.0	1000.0
1250	1258.9	1259.9
1600	1584.9	1587.4
2000	1995.2	2000.0
2500	2511.9	2519.8
3150	3162.3	3174.8
$\vdots$	$\vdots$	$\vdots$

Table 1: Preferred frequencies for third octave bands in Hz.

Example 1: m-band spectrum of white noise

A stochastic signal with a constant power spectral density $\hat{C}_{pp}(\omega) = const =: P_0$ is called *white noise* (german: "weisses Rauschen"). The name is referring to light, which appears white when all colors (frequencies) are represented with equal intensity. From eqn(27) one has: $\tilde{p}_i^2/p_{\text{ref}}^2 = P^*(2^{m/2} - 2^{-m/2})f_i^c$. where $P^* := P_0 f_0^c / p_{\text{ref}}^2$. With $f_i^c = 2^{mi} f_0^c$ the corresponding band levels are $L_{pm}^i = 10\{\lg[P^*(2^{m/2} - 2^{-m/2})] + im \lg 2\}$. This shows that the octave band level ($m = 1/1$) of white noise increases linearly by 3 dB/octave and that the third octave band level ($m = 1/3$) increases linearly by 1 dB/third octave. Note that trivially, the narrow band spectrum of white noise is constant.

Example 2: m-band spectrum of pink noise

Pink noise (german: "rosa Rauschen") is defined as a stochastic signal with constant m-band level. What does the power spectral density of pink noise look like? Taking the definition we have $\tilde{p}_i^2 = 2\int_a^b \hat{C}_{pp}(f)\; df = const$ with $a := f_i^l = 2^{-m/2} f_i^c$ and $b := f_i^u = 2^{m/2} f_i^c$. Differentiation w.r.t. f_i^c yields the determining equation for $\hat{C}_{pp}$: $\dfrac{\partial \tilde{p}_i^2}{\partial f_i^c} = 0 = \dfrac{\partial b}{\partial f_i^c}\hat{C}_{pp}(b) - \dfrac{\partial a}{\partial f_i^c}\hat{C}_{pp}(a)$. This is equal to $\dfrac{\hat{C}_{pp}(2^{m/2} f_i^c)}{\hat{C}_{pp}(2^{-m/2} f_i^c)} = 2^{-m}$, showing that the power spectral density has the form $\hat{C}_{pp}(f) = \dfrac{C_1}{f}$. The integration constant may be determined to be $C_1 = const/(m \ln 4)$. White light becomes "redish" when the power spectral density is lifted to higher values in the lower frequency range (red light). The above result for $\hat{C}_{pp}$ shows indeed that this is the case.

Example 3: comparison of narrow band and 1/3 octave band spectrum of some noise signal

A general noise signal consisting of a superposition of broadband and tonal components is considered. Its narrow band spectrum as depicted in the left of figure 8. Up to a frequency of about

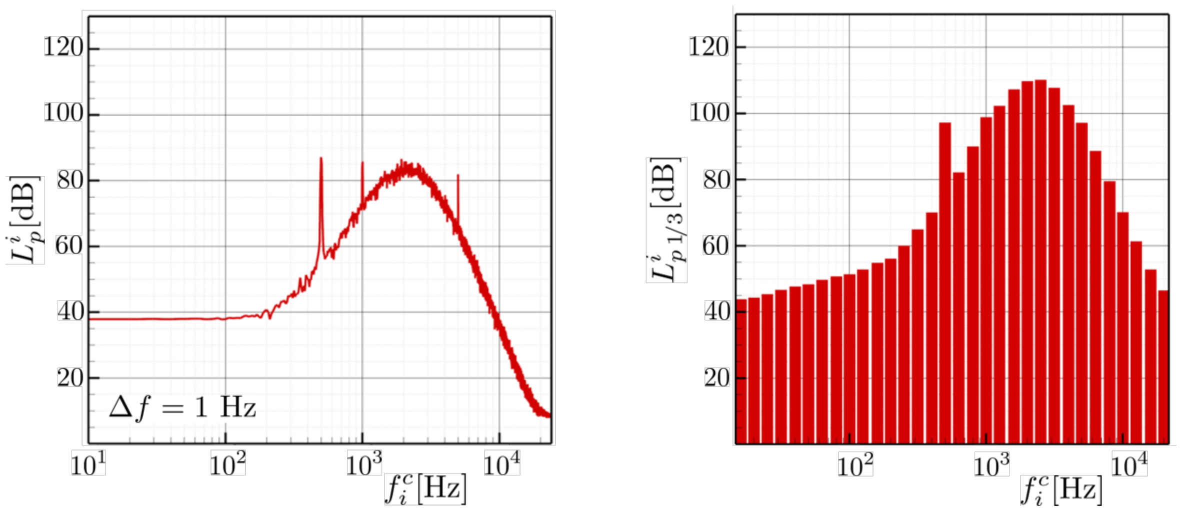

Figure 8: left: narrow band spectrum of a general signal, right: corresponding third octave band spectrum.

200Hz the signal displays a white noise type (i.e. constant) narrow band spectrum. Sharp peaks at 500Hz, 1000Hz and 5000Hz indicate that tones are superimposed to the broadband spectrum. When looking at the third octave band spectrum of the same signal (right of figure *8*), one notices that the band levels increase at constant rate below 200Hz, showing that white noise increases with 1dB/third octave band. While the strong tone at 500Hz is clearly seen in the 500Hz band, the other tones get buried in the wide integration bands and can no longer be identified. Third octave band spectra generally pronounce the higher frequency range.

2.1.5.4 Loudness and A-weighting of complex signals, noisiness and perceived noise level

When considering noise from machines usually one has to deal with complex signals quite different from sinusoidal behavior. Therefore the above introduced definition of loudness level (2.1.5.1) is insufficient for most situations. There are several approaches to generalize the desired adaptation of the sound pressure level to hearing sensitivity.

Loudness of complex signals

Now that we can determine the frequency content of complex signals in band levels it becomes possible to generalize the defintion of loudness and loudness level as introduced for pure tone signals in section 2.1.5.1:

i) the signal is analysed into its (third) octave band spectrum,
ii) then for the centre frequency f_i^c and the corresponding band level $L_{p1/1}(f_i^c)$ (or $L_{p1/3}(f_i^c)$) of each band a loudness value s_i is determined from diagram *4*,
iii) finally an "overall loudness" S is determined from

$$S := s_{max} + C\left[\sum_{bands} s_i - s_{max}\right]. \tag{28}$$

where $C = 0.3$ for octave bands and $C = 0.15$ for third octave bands and s_{max} denotes the largest loudness level among all bands. The overall loudness is used to determine the overall loudness level L_L in phons in analogy to eqn (17):

$$L_L := 40 + 33.2 \lg S, \quad [L_L] = \text{phon} \tag{29}$$

A-weighting of complex signals
A simplified and often used approach to take into account the sensitivity of hearing is the weighting of the physical (also called "linear") spectrum L_p^i of the complex signal in analogy to section 2.1.5.2. The most commonly used weighting is based on the A-curve (see figure 5). Knowing the physical ("linear") band level L_p^i of each band i of our complex signal, eqn (27), the A-weighting delta $\Delta L_A(f_i^c)$ of the band's centre frequency f_c^i is added to it, just as the weighting of tones in (18):

$$L_{pA}^i = L_p^i + \Delta L_A(f_i^c).$$

In order to determine the *overall A-weighted sound pressure level* L_{pA} of the complex signal the A-weighted mean square values $\tilde{p}_{iA}^2 = 10^{L_{pA}^i/10}\ p_{\text{ref}}^2$ of all bands i are summed up (energetic sum):

$$\tilde{p}_A^2 = \sum_{i=-\infty}^{\infty} \tilde{p}_{iA}^2 \Rightarrow L_{pA} = 10 \lg \left(\frac{\tilde{p}_A^2}{p_{\text{ref}}^2} \right) \text{dB(A)} = 10 \lg \left\{ \sum_{i=-\infty}^{\infty} 10^{(L_p^i + \Delta L_A(f_i^c))/10} \right\} \text{dB(A)}$$

In the same way the noise weighting is done for the intensity and -more importantly- for the sound power $P \to P_A$ or rather the *A-weighted sound power level* $L_{WA} = 10 \lg(P_A/P_{\text{ref}})$dB(A). Typically the value of L_{WA} is found on a label on technical devices, setting out their compliance to respective noise certification limits.

Noisiness and Perceived Noise Level
Another quantity for the description of noise, especially aviation noise, is the so called *noisiness* n (german: "Lärmigkeit"). Analogously to the loudness (for pure tone perception) equal noisiness curves have been generated based on tests with a large number of persons.

The noisiness of a single third octave band noise signal (of diffuse incidence) with fixed centre frequency f_i^c and given band level $L_{p1/3}(f_i^c)$ is the factor by which this signal is perceived louder than a respective 1000 Hz third octave band noise signal of band level $L_{p1/3}(1000Hz) = 40dB$ (see diagram 9). The unit of the noisiness is called "noy" and the value for the noisiness of the reference third octave band signal of $L_{p1/3}(1000Hz) = 40$ dB is defined to be $n = 1$ noy.

In general noise does not consist of only one third octave band noise signal, but is rather a mixture of the contribution of very many third octave bands. In fact, the third octave band spectrum tells us about the magnitude ($L_{p1/3}$) of all these contributions. In order to arrive at a perceived noise level for such a complex noise signal the noisiness values of all the third octave bands are determined and integrated into one single overall noisness value N (compare with eqn (28) for the loudness):

$$N := n_{max} + 0.15 \left[\sum_{all1/3oct.} n_i - n_{max} \right] \tag{30}$$

where n_{max} is the largest noisiness value occuring in the signal. The overall noisiness serves to define the so called *perceived noise level* L_{PN} (or PNL), which is the analogy to the loudness level L_L (or L_N), based on the noisiness rather than on the loudness.

$$L_{PN} = 40 + 33.2 \lg N, \quad [L_{PN}] = \text{PNdB} \tag{31}$$

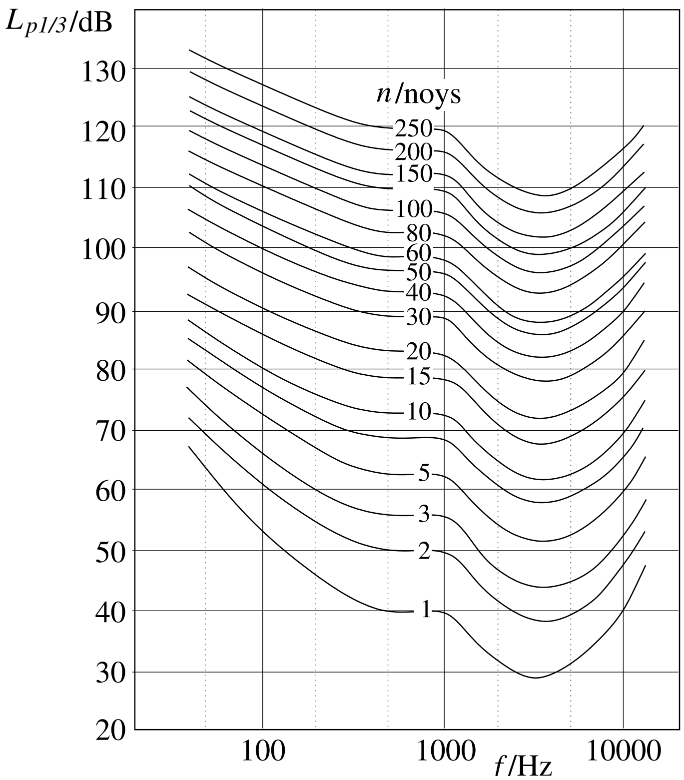

Figure 9: Noisiness contours n =const as a function of band frequency and third octave band level.

2.2 Basic equations

In this section we begin with deriving the governing equations of acoustic phenomena. We will discuss simple solutions in order to explain the physical mechanisms which determine the sound near and far from sources. Then we introduce some mathematical tools, which are very helpful for handling acoustic problems theoretically and for the understanding of the literature. Lastly we define the multipole expansion of acoustic sources explain the concepts behind it and end the section with the reciprocity relation. All this will be derived for a non-moving medium, although the derivation of the wave equation really starts with the general conservation equations for moving media.

The dynamics of a compressible fluid is described by the Navier-Stokes equations (conservation of mass, momentum and energy):

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = \dot{m} \tag{32}$$

$$\frac{\partial\rho\boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}\boldsymbol{v}) + \boldsymbol{\nabla}p = \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \boldsymbol{f} + \dot{m}\boldsymbol{v} \tag{33}$$

$$\frac{\partial\rho e_t}{\partial t} + \boldsymbol{\nabla}\cdot(\rho e_t\boldsymbol{v}) + \boldsymbol{\nabla}\cdot(p\boldsymbol{v}) = -\boldsymbol{\nabla}\cdot\boldsymbol{q} + \boldsymbol{\nabla}\cdot(\boldsymbol{\tau}\boldsymbol{v}) + \dot{\vartheta} + \boldsymbol{f}\cdot\boldsymbol{v} + \dot{m}e_t \tag{34}$$

Where ρ represents the fluid density, $\boldsymbol{v}$ its velocity vector, p the pressure, $e_t = e + \frac{1}{2}\boldsymbol{v}^2$ the specific total energy, made up of specific internal energy e and the specific kinetic energy. $\boldsymbol{\tau}(\boldsymbol{v},\mu)$ denotes the friction-related stress tensor which depends on the (usually temperature T-dependent) dynamic viscosity $\mu(T)$; typically for a Newtonian medium under Stokes'es hypothesis $\boldsymbol{\tau} = \mu(\boldsymbol{\nabla}\boldsymbol{v} + {}^t\boldsymbol{\nabla}\boldsymbol{v} - \frac{2}{3}\boldsymbol{I}\boldsymbol{\nabla}\cdot\boldsymbol{v})$. The heat flux vector is denoted $\boldsymbol{q}(T,k)$ and depends on the heat conductivity $k(T)$ of the medium; typically, for Fourier's law of heat conduction $\boldsymbol{q} = -k\boldsymbol{\nabla}T$.

We have also introduced (given) hypothetical independent source terms for mass $\dot{m}$, external forces $\boldsymbol{f}$ and heat $\dot{\vartheta}$. Since the additional mass is part of the flowing medium it has to assume its local velocity and internal energy, such that respective contributions to the momentum ($\dot{m}\boldsymbol{v}$) and the total energy ($e_t\boldsymbol{v}$) are to be accounted for in the equations. Although mass cannot be created in a setting of classical mechanics (i.e. $\dot{m} = 0$), it becomes reasonable when certain processes cannot be resolved in actual computations, e.g. if one would like to consider the radial mass injection of very fine jets into a long pipe system, which one desires to treat one-dimensionally. Then the mass ocurring would be injected along a space dimension, which is not part of the computational equations. In such a setting an appropriate modelling of this mass injection would indeed give a non-zero $\dot{m}$. In the same way, mass and momentum injection processes in e.g. an aero-engine may be lumped together and accounted for in a so called actuator disk. One may also think of $\dot{m}$ as of representing the fluid-displacement of a differential element of a rigid body in the flow. Such a view is justified when considering the flow variables as generalized functions, who allow for an explicit transfer of boundary conditions as (usually singular) source terms into the differential equation. We will come back to such a formulation later. For current purposes we just allow for some (given) terms on the right hand side of our equations and call them sources.

The system (32-34) along with the mentioned expressions for τ, μ, $\boldsymbol{q}$ and k is not closed yet: we have the 7 unknowns $\rho, \boldsymbol{v}, e_t$ (or equivalently e), p and T but only 5 equations. In order to close the system we further need 2 relations: the thermal and the caloric state equation for the fluid.

For the remainder of the script we assume a fluid in thermodynamic equilibrium, i.e.

$$\text{thermal: } \rho = \rho(T, p) \quad \text{calorical: } e = e(T, p) \tag{35}$$

The above form of the equations is called conservative form, because the equations represent the integrands of the conservation balance integrals. The source terms are most easily introduced here. The injected mass occurs in the momentum equations since the fluid has to exert a force to accelerate the mass to the ambient velocity $\boldsymbol{v}$. The mass flux also appears in the energy equation. Likewise the force component in or against the flow direction of the applied external force $\boldsymbol{f}$ contributes to the power balance.

We take the above equations as reference, but in acoustics we are more interested in a simpler formulation. Especially we like to involve the entropy s as a variable. We obtain the so called primitive formulation in the following way: i) multiply the mass balance (32) by $\boldsymbol{v}$ and subtract from momentum balance (33); ii) take the dot product of the momentum balance (33) with $\boldsymbol{v}$ and subtract from energy balance (34). Also multiply (32) by e and subtract from (34). Finally we have:

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{v} + \dot{m} \tag{36}$$

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \boldsymbol{f} \tag{37}$$

$$\rho \frac{De}{Dt} = -p \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{\tau} : \boldsymbol{\nabla} \boldsymbol{v} - \boldsymbol{\nabla} \cdot \boldsymbol{q} + \dot{\vartheta} \tag{38}$$

In which $\frac{D}{Dt} := \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ denotes the substantial or material derivative, i.e. the temporal change felt by the fluid element in motion, passing position $\boldsymbol{x}$ at time t. As will be seen subsequentially, it is convenient to re-formulate the energy balance (38) into the entropy equation. Its definition is (Gibbs)

$$T \delta s = \delta e - (p/\rho^2)\, \delta \rho = \delta h - (1/\rho)\, \delta p \tag{39}$$

with T the temperature and h the specific enthalpy. The symbol δ denotes here a total differential in the sense of equilibrium thermodynamics due to which a variation of one thermodynamic variable is completely determined in terms of the variation of two other thermodynamic variables all taken at the same point in time and space. Is the fluid globally in thermodynamic equilibrium (which we will further assume), then it does not matter, how the change δ comes about. Of course the same kind of variation has to be considered for all variables, may it be due to a separate temporal change ($\delta = dt \cdot \partial/\partial t$) or a separate spatial change ($\delta = dx \cdot \partial/\partial x$). If the equilibrium is only local, i.e. only within the size of a fluid element, then this couples the temporal and spatial variations of the variables and $\delta = dt \cdot \frac{D}{Dt}$ must be understood as a material change, i.e. according to some D/Dt.

Now we go on in replacing the internal energy in equation (38) by use of equation (39). In the most general case δ may be replaced by $\delta = dt \cdot \frac{D}{Dt}$, i.e. the material change following a fluid particle. If we do so in (39) and multiply by $\rho/(T dt)$ the resulting r.h.s-terms $\rho \frac{De}{Dt}$ and $\frac{D\rho}{Dt}$ can be substituted from (36) and (38) to give the entropy equation

$$\rho \frac{Ds}{Dt} = \frac{1}{T} \left[\boldsymbol{\tau} : \boldsymbol{\nabla} \boldsymbol{v} - \boldsymbol{\nabla} \cdot \boldsymbol{q} + \dot{\vartheta} - \dot{m} \frac{p}{\rho} \right] \tag{40}$$

We note that the term $\Phi := \boldsymbol{\tau}\boldsymbol{:}\boldsymbol{\nabla v}$ is called "dissipation function", of which it can be shown that it is never negative[4]. In other words, the effect of viscous friction is strictly to increase the entropy in a fluid element and thus make the process irreversible. As expected, heat conduction may increase or decrease entropy depending on the direction of the flux vector, and the heat sources themselves directly act as to change entropy. Also the addition of mass changes the entropy.

Upon introducing the entropy as a new variable, we have eliminated the internal energy e. In order to close our equation system we now replace the caloric equation of state (35) by a thermodynamic relation between variables of state, in this case certainly $\rho = \rho(p, s)$, which translates into

$$\delta\rho = \underbrace{\left(\frac{\partial\rho}{\partial p}\right)_s}_{=:\,1/a^2} \delta p + \underbrace{\left(\frac{\partial\rho}{\partial s}\right)_p}_{=:\,-\rho\sigma} \delta s \tag{41}$$

where we have introduced the definitions

$$a^2 := \left(\frac{\partial\rho}{\partial p}\right)_s^{-1} \quad ; \qquad \sigma := -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_p \tag{42}$$

Note that it is reasonable to introduce an expression a^2, i.e. a strictly positive quantity because a pressure increase $\partial p > 0$ will necessarily result in a compression, meaning an increase in the density $\partial\rho > 0$ and vice versa. We want to use (41) to formulate an evolution equation for the pressure. For this case we again consider the most general case of a variation δ in the thermodynamic relation (41), i.e. we look at changes following a fluid particle and replace $\delta = dt \cdot \frac{D}{Dt}$. Then we divide by dt, which finally leaves our desired relation for the pressure

$$\frac{1}{a^2}\frac{Dp}{Dt} = \frac{D\rho}{Dt} + \sigma\rho\frac{Ds}{Dt} \tag{43}$$

which explicitly states that for isentropic flows (i.e. if $\frac{Ds}{Dt} = 0$) the changes of pressure and density along the particle paths are directly coupled by the scalar a^2. Note that this does not mean then that the entropy is constant everywhere. This would be the case, if on a surface upstream, transverse to the flow, the entropy was constant for all times. From (43) it is seen explicitly, why it was advantageous to consider material changes $\frac{D}{Dt}$. The reason is, we may now insert the entropy from (40) and the density from (36) into our pressure equation to obtain:

$$\frac{1}{a^2}\frac{Dp}{Dt} = -\rho\boldsymbol{\nabla\cdot v} + \frac{\sigma}{T}(\boldsymbol{\tau} : \boldsymbol{\nabla v} - \boldsymbol{\nabla\cdot q} + \dot{\vartheta}) + \dot{m}\Big(1 - \frac{\sigma}{T}\frac{p}{\rho}\Big) \tag{44}$$

Note, that so far we have nothing said about the fluid, i.e. we have not specified a special thermal equation of state. The set of equations (36, 37, 44) constitutes the governing equations for density, velocity and pressure. Considering a^2 and σ to be given properties of the fluid, then we only need to specify the thermal equation of state $\rho(T, p)$ which closes our equation system. In what follows, we will often refer to perfect gases, i.e. gases with the following thermal and calorical properties:

$$\rho = \frac{p}{RT} \tag{45}$$

[4]This is due to $\Phi = \mu(\boldsymbol{\nabla v} + {}^t\boldsymbol{\nabla v} - \frac{2}{3}\boldsymbol{I\nabla\cdot v})\boldsymbol{:}\ \boldsymbol{\nabla v} = \mu([\boldsymbol{\nabla v} - \frac{1}{3}\boldsymbol{I\nabla\cdot v}] + [{}^t\boldsymbol{\nabla v} - \frac{1}{3}\boldsymbol{I\nabla\cdot v}])\boldsymbol{:}(\boldsymbol{\nabla v} - \frac{1}{3}\boldsymbol{I\nabla\cdot v}) + \mu(\boldsymbol{\nabla v} + {}^t\boldsymbol{\nabla v} - \frac{2}{3}\boldsymbol{I\nabla\cdot v})\boldsymbol{:I}\frac{1}{3}\boldsymbol{\nabla\cdot v}$. From this: $\Phi = \frac{1}{2}\mu(\boldsymbol{\nabla v} + {}^t\boldsymbol{\nabla v} - \frac{1}{3}\boldsymbol{I\nabla\cdot v})^2 + \mu(\frac{2}{3}\boldsymbol{\nabla\cdot v})^2 \geq 0$.

and

$$de = c_v dT \tag{46}$$

where R denotes the specific gas constant and c_v the (constant) specific heat capacity for constant volume of the fluid. We also introduce the specific heat capacity for constant pressure $c_p = R + c_v$ and the isentropic coefficient $\gamma = c_p/c_v$ with e.g. $\gamma \approx 1.4$ for air. Upon using (45), (46) and (39) we can now express everything in terms of pressure and density, i.e.

$$a^2 = \gamma RT = \gamma p/\rho \tag{47}$$

$$\sigma = 1/c_p \tag{48}$$

which closes our system (36, 37, 44). In summary, our final equation system reads

$$\begin{aligned} \frac{D\rho}{Dt} + \rho \boldsymbol{\nabla \cdot v} &= \dot{m} \\ \rho \frac{D\boldsymbol{v}}{Dt} + \boldsymbol{\nabla} p - \boldsymbol{\nabla \cdot \tau} &= \boldsymbol{f} \qquad (49) \\ \frac{1}{a^2}\frac{Dp}{Dt} + \rho \boldsymbol{\nabla \cdot v} - \frac{\sigma}{T}(\boldsymbol{\tau : \nabla v} - \boldsymbol{\nabla \cdot q}) &= \frac{\sigma}{T}\dot{\vartheta} + \left(1 - \sigma \frac{p}{\rho T}\right)\dot{m} \qquad (44) \end{aligned}$$

For a perfect gas, T is eliminated according to the thermal equation of state (45) while a^2 and σ follow (47) and (48) respectively. For a fluid different from a perfect gas (44) is supplemented by some properties (35) of which immediately follow $a^2(p, T)$ and $\sigma(p, T)$. For instance, an incompressible fluid could be characterized by $\rho = \rho(T)$ and thus $(\partial/\partial p)\rho = a^{-2} = 0$ as well as $c_v = c_p$. Note that in this case the divergence $\boldsymbol{\nabla \cdot v}$ is determined through external heating $\dot{\vartheta}$ and mass sources $\dot{m}$. The distinction between incompressible and compressible behaviour of the pressure will subsequentially play an important role when characterizing sound.

Generally, relative density gradients, i.e. compressibility effects, occur in a medium, when the flow acceleration, i.e. the material derivative of the flow velocity becomes high compared to the square of the speed of sound. For steady flow about some aerodynamic object this is the case whenever the flow has high subsonic freestream speeds. If on the other hand the flow is strongly unsteady (high frequency) then the acceleration may be dominated by the time derivative of the speed, which may be high even at arbitrary small flow speeds. In this case one is typically dealing with sound. The fundamental difference of compressible and incompressible behaviour at high temporal gradients (but for small changes in space is illustrated in figure *1*. The left part of this figure shows two snapshots of the motion of an incompressible medium as a result of an oscillating sphere. In contrast, the motion of a compressible medium, excited by the idential motion is shown in the right part of the figure. Note, that very close to the sphere, the motion of the surrounding particles is very similar though.

2.3 Linear equations of gas dynamics

What is our objective? We want to predict the acoustic variable at the listener's position. It is the pressure fluctuation p' deviating temporally from the mean ambient pressure p^0, which we sense with the ear (i.e. $p = p^0 + p'$). So, our aim is certainly to derive appropriate equations, describing the p'-field or at least a field, of which p' is simply (algebraically) extractable at positions where the value of the sound pressure is needed.

We need to describe small perturbations of the pressure about its mean value. More generally, we consider all our flow variables $\boldsymbol{V}(\boldsymbol{x}, t) = (\rho, \boldsymbol{v}, p, a^2, \sigma, \ldots)$ as being composed of a mean steady value $\boldsymbol{V}^0(\boldsymbol{x}) = (\rho^0, \boldsymbol{v}^0, p^0, (a^2)^0, \sigma^0, \ldots)$ plus a small, but unsteady perturbation $\boldsymbol{V}'(\boldsymbol{x}, t) = (\rho', \boldsymbol{v}', p', (a^2)', \sigma', \ldots)$. The smallness of the perturbation is expressed by introducing a small number $\epsilon \ll 1$:

$$(\rho, \boldsymbol{v}, p, a^2, \sigma, \cdots) = (\rho^0 + \epsilon\rho', \boldsymbol{v}^0 + \epsilon\boldsymbol{v}', p^0 + \epsilon p', (a^2)^0 + \epsilon(a^2)', \sigma^0 + \epsilon\sigma', \ldots) \tag{50}$$

We assume our mean flow satisfies the steady form of (36, 49, 44) without any sources, i.e. $\dot{m}^0 = 0$, $\boldsymbol{f}^0 = \boldsymbol{0}$, $\dot{\vartheta}^0 = 0$. Then

$$\boldsymbol{v}^0 \cdot \boldsymbol{\nabla}\rho^0 = -\rho^0 \boldsymbol{\nabla}\cdot\boldsymbol{v}^0 \tag{51}$$

$$\rho^0 \boldsymbol{v}^0 \cdot \boldsymbol{\nabla}\boldsymbol{v}^0 - \boldsymbol{\nabla}\cdot\boldsymbol{\tau}^0 = -\boldsymbol{\nabla}p^0 \tag{52}$$

$$\frac{1}{(a^2)^0}\, \boldsymbol{v}^0 \cdot \boldsymbol{\nabla}p^0 = -\rho^0 \boldsymbol{\nabla}\cdot\boldsymbol{v}^0 + \frac{\sigma^0}{T^0}(\boldsymbol{\tau}^0 : \boldsymbol{\nabla}\boldsymbol{v}^0 - \boldsymbol{\nabla}\cdot\boldsymbol{q}^0) \tag{53}$$

According to standard perturbation techniques the equations, describing the dynamics of the first order perturbations are obtained by i) inserting (50) into (36, 49, 44), ii) differentiating with respect to ϵ and iii) letting $\epsilon \to 0$. This gives

$$\frac{D^0\rho'}{Dt} + \rho^0 \boldsymbol{\nabla}\cdot\boldsymbol{v}' + \boldsymbol{v}'\cdot\boldsymbol{\nabla}\rho^0 + \rho'\boldsymbol{\nabla}\cdot\boldsymbol{v}^0 = \dot{m}' \tag{54}$$

$$\rho^0 \frac{D^0\boldsymbol{v}'}{Dt} + \boldsymbol{\nabla}p' + \rho^0\boldsymbol{v}'\cdot\boldsymbol{\nabla}\boldsymbol{v}^0 + \rho'\boldsymbol{v}^0\cdot\boldsymbol{\nabla}\boldsymbol{v}^0 = \boldsymbol{f}' \tag{55}$$

$$\underbrace{\frac{1}{(a^2)^0}}_{\left[\frac{\rho^0}{\gamma p^0}\right]_p}\left(\frac{D^0p'}{Dt} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}p^0\right) + \rho^0\boldsymbol{\nabla}\cdot\boldsymbol{v}' + \underbrace{\left(\rho^0\frac{(a^2)'}{(a^2)^0} + \rho'\right)}_{\left[\frac{\rho^0}{p^0}\,p'\right]_p}\boldsymbol{\nabla}\cdot\boldsymbol{v}^0 = \underbrace{\frac{\sigma^0}{T^0}\dot{\vartheta}' + \left(1 - \frac{\sigma^0 p^0}{\rho^0 T^0}\right)\dot{m}'}_{\left[\frac{\gamma-1}{\gamma}\frac{\rho^0}{p^0}\dot{\vartheta}' + \frac{1}{\gamma}\dot{m}'\right]_p} =: \dot{\theta}' \tag{56}$$

where the expression $\frac{D^0}{Dt} := \frac{\partial}{\partial t} + \boldsymbol{v}^0\cdot\boldsymbol{\nabla}$ denotes the time derivative along the streamlines of the mean flow. In this system the perturbations of viscous stresses $\boldsymbol{\nabla}\cdot\boldsymbol{\tau}'$ in (55), viscous dissipation and heat conduction $[\frac{\sigma}{T}(\boldsymbol{\tau} : \boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla}\cdot\boldsymbol{q})]'$ in (56) have been omitted, since these are typically very small quantities and relevant in very rare cases only. The brackets and the subscript p indicate the specialization of (56) for a perfect gas using (47) and (48). For the special case of an incompressible medium $(a^2)^0 \to \infty$ the pressure equation degenerates into an equation for the divergence of the mass flux $(\rho \boldsymbol{v})'$ by external mass addition and heating. The pressure equation (56) shows explicitly that first assuming a perfect gas (see squared brackets) with $(a^2)^0 = \gamma p^0/\rho^0$ and then letting $(a^2)^0 \to \infty$ would have yielded a completely different (errorneous) result.

We note that in case of resonances, excited by the sources, these linear equations are not uniformly valid (i.e. valid only over a finite time of observation). With resonances we specifically mean so called "absolute" or "global hydrodynamic instabilities", the study of which by its own is a whole discipline of fluid mechanics. For appropriate flow parameters, absolute instabilities occur for instance in the wake flow behind blunt bodies. They initiate the typical von Karman vortex street, which is a self-sustained flow oscillation. Again, we exclude such phenomena when dealing with our equation system.

Our set of equations bears the name "linearized gas dynamic equations" and according to our derivation they describe small perturbations of a steady mean flow (although they would equally be valid for a time varying mean flow). For an inviscid flow this system is exact, while for general viscous flows it assumes the explicit neglect of the perturbations of viscous stress, dissipation and heat conduction. More often this set of equations is called "Linearized Euler Equations (LEE)", although, again, it is not only used for strictly inviscid, but also viscous mean flows. In particular, the dynamics of sound waves propagating in the mean flow field is included. A considerable work of aeroacousticians has been, to derive a separate equation for the acoustic pressure perturbation or some other appropriate acoustic variable from this set of equations. Such an equation would certainly be some sort of a wave equation, since we know that in acoustics we are dealing with "sound waves". The derivation of wave equations is what we will do in the following sections as well. We will start with the simplest case, i.e. non-moving fluid and look at flow effects when introducing some simple non-zero flow fields.

2.4 Acoustics in stagnant homogeneous media

For the beginning we consider the case $\boldsymbol{v}^0 = \boldsymbol{0}$, e.g. a medium at rest. Note, that due to (52) this implies $p^0 \neq p^0(\boldsymbol{x})$, i.e. $p^0 = const =: p_\infty$, while the density ρ^0 may still be a function of space. Let us re-write (54-56) for this case

$$\frac{\partial \rho'}{\partial t} + \boldsymbol{v}' \cdot \boldsymbol{\nabla} \rho^0 + \rho^0 \boldsymbol{\nabla} \cdot \boldsymbol{v}' = \dot{m}' \tag{57}$$

$$\rho^0 \frac{\partial \boldsymbol{v}'}{\partial t} + \boldsymbol{\nabla} p' = \boldsymbol{f}' \tag{58}$$

$$\frac{1}{(a^2)^0} \frac{\partial p'}{\partial t} + \rho^0 \boldsymbol{\nabla} \cdot \boldsymbol{v}' = \dot{\theta}' \tag{59}$$

2.4.1 Intensity and power

With the derivation of the perturbation equations for a non-moving medium we now have the means to show how the sound intensity and sound power in (9) and (12) are linked with the sources. First we multiply (58) by $\boldsymbol{v}'$ to obtain

$$\frac{\partial}{\partial t} \left(\rho^0 \tfrac{1}{2} \boldsymbol{v}'^2 \right) + \boldsymbol{v}' \cdot \boldsymbol{\nabla} p' = \boldsymbol{v}' \cdot \boldsymbol{f}'$$

Next we multiply (59) by p'/ρ^0 and obtain

$$\frac{\partial}{\partial t}\left(\frac{\frac{1}{2}p'^2}{\rho^0(a^2)^0}\right) + p'\boldsymbol{\nabla}\boldsymbol{\cdot}v' = \frac{1}{\rho^0}\dot{\theta}'p'$$

Upon adding these two equations we find

$$\frac{\partial}{\partial t}\Big(\underbrace{\rho^0\tfrac{1}{2}\boldsymbol{v}'^2 + \tfrac{1}{2}\frac{p'^2}{\rho^0(a^2)^0}}_{=:\,E}\Big) + \boldsymbol{\nabla}\boldsymbol{\cdot}(p'\boldsymbol{v}') = \underbrace{\boldsymbol{v}'\boldsymbol{\cdot}\boldsymbol{f}' + \frac{1}{\rho^0}\dot{\theta}'p'}_{=:\,Q},$$

where we call E the energy density. Note that if we take the time average of this equation due to (2) the energy density term drops out because the temporal mean of a time derivative is zero by definition. Moreover, the expression in the divergence operator apears to be the acoustic intensity $\boldsymbol{I}$. One may now integrate the above equations over some control volume V, which is to contain all the sources, i.e.

$$\int\limits_V \boldsymbol{\nabla}\boldsymbol{\cdot}(\overline{p'\boldsymbol{v}'})\,dV = \int\limits_V \overline{Q}\,dV \Longleftrightarrow \oint\limits_{\partial V}(\overline{p'\boldsymbol{v}}')\boldsymbol{\cdot}\boldsymbol{n}\,dA = \int\limits_{V_S}\overline{Q}\,dV$$

where we have used Gauss's theorem to convert the volume integral on the left hand side into a surface integral. Further, $V_S \in V$ denotes the (sub-) volume, within which the sources ($\boldsymbol{f}'$ or $\dot{\theta}'$) are non-zero. Finally we have

$$0 + \underbrace{\oint\limits_{\partial V}\boldsymbol{I}\boldsymbol{\cdot}\boldsymbol{n}\,dA}_{=\,P} = \int\limits_{V_S}\overline{Q}\,dV.$$

This last relation shows that the sound power P is independent of the chosen control volume V, as long as all sources are contained inside this volume. We need to recall though that in deriving the above relation we have neglected dissipation and heat conduction. In order to assess these effects we can think of e.g. $\boldsymbol{f}'$ as of incorporating the actual source as well as a second force contribution due to the viscous friction $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau}'$. Likewise one would have incorporated the heat conduction term in the pressure equation $(\frac{\sigma}{T}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q})'$ in the heat source $\dot{\theta}'$. The source Q in the above sound power balance would have obtained an extra term $Q_{diss} = (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau}')\boldsymbol{\cdot}\boldsymbol{v}' + \frac{1}{\rho^0}(\frac{\sigma}{T}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q})'p'$. Since friction and heat conduction occurs everywhere inside V the extra contribution Q_{diss} would then be non-zero everywhere inside the control volume V. In this case the sound power P would indeed depend on the choice of the control volume and not be a conserved quantity. Note that all signals not originating from within V give no contribution to the sound power P. This result is quite important practically for the measurement of the noise output of an object in a noisy environment.

2.4.2 Acoustic wave equations

2.4.2.1 Sound field equation for pressure. We may now easily isolate a pressure equation from the linear gas dynamic equations in the following way: i) divide the linearized momentum equation (58) by ρ^0 and take the divergence to get

$$\frac{\partial \boldsymbol{\nabla}\cdot\boldsymbol{v}'}{\partial t} = -\boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{\nabla}p'\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{f}'\right)$$

then (ii) take the time derivate of the pressure equation (59), i.e.

$$\frac{1}{(a^2)^0}\frac{\partial^2 p'}{\partial t^2} + \rho^0\frac{\partial \boldsymbol{\nabla}\cdot\boldsymbol{v}'}{\partial t} = \left(1 - \frac{\sigma^0 p_\infty}{\rho^0 T^0}\right)\frac{\partial \dot{m}'}{\partial t} + \frac{\sigma^0}{T^0}\frac{\partial \dot{\vartheta}'}{\partial t}$$

It is obvious that we can now eliminate the velocity perturbations $\boldsymbol{v}'$ from these two relations to finally obtain

$$\begin{aligned} \frac{1}{(a^2)^0}\frac{\partial^2 p'}{\partial t^2} - \rho^0\boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{\nabla}p'\right) &= Q_p \\ Q_p &= \underbrace{\left(1 - \frac{\sigma^0 p_\infty}{\rho^0 T^0}\right)\frac{\partial \dot{m}'}{\partial t} + \frac{\sigma^0}{T^0}\frac{\partial \dot{\vartheta}'}{\partial t}}_{\left[\frac{\gamma-1}{(a^2)^0}\frac{\partial \dot{\vartheta}'}{\partial t} + \frac{1}{\gamma}\frac{\partial \dot{m}'}{\partial t}\right]_{pg}} - \rho^0\boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{f}'\right) \end{aligned} \qquad (60)$$

This is the wave equation for the pressure perturbation p' in a stagnant medium of variable mean density $\rho^0(\boldsymbol{x})$. The left hand side describes the wave dynamics (and is therefore called wave operator) while the right hand side describes the given source functions, which alltogether we call Q_p, a given function. Again, we have indicated the appearance of the r.h.s.-terms with a subscript pg in case of a perfect gas. It is noteworthy that the wave operator depends solely on ρ^0 and $(a^2)^0$, i.e. these two quantities define the wave dynamics of the pressure waves. For a perfect gas the wave equation (60) further simplifies since by (47) we have $\rho^0 = \gamma p_\infty/(a^2)^0$. Therefore, substituting ρ^0 in (60) for a perfect gas gives

$$\frac{\partial^2 p'}{\partial t^2} - \boldsymbol{\nabla}\cdot[(a^2)^0\boldsymbol{\nabla}p'] = \gamma^{-1}(a^2)^0\frac{\partial \dot{m}'}{\partial t} + (\gamma-1)\frac{\partial \dot{\vartheta}'}{\partial t} - \boldsymbol{\nabla}\cdot[(a^2)^0\boldsymbol{f}']$$

This equation shows the perturbation dynamics of the pressure (l.h.s.) to be completely described by the quantity $(a^2)^0$ for a perfect gas. It is obviously of fundamental physical importance for the understanding of sound waves.

In order to interpret $(a^2)^0$ physically, let us go back to (60) and consider the most simple case, i.e. a constant mean density $\rho^0 = const =: \rho_\infty$ of the medium and thus $(a^2)^0 = const =: (a^2)_\infty$. Then

$$\frac{1}{(a^2)_\infty}\frac{\partial^2 p'}{\partial t^2} - \Delta p' = Q_p \tag{61}$$

$$Q_p = \underbrace{\left(1 - \frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty}\right)\frac{\partial \dot{m}'}{\partial t} + \frac{\sigma_\infty}{T_\infty}\frac{\partial \dot{\vartheta}'}{\partial t}}_{\left[\frac{1}{\gamma}\frac{\partial \dot{m}'}{\partial t} + \frac{\gamma - 1}{(a^2)_\infty}\frac{\partial \dot{\vartheta}'}{\partial t}\right]_{pg}} - \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{f}'$$

where $\Delta = \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}$ denotes the Laplacian. Before solving this equation let us have a first look at the source term Q_p.

Without actually solving the acoustic wave equation (60) for the sound pressure p' due to the given sources some statements about the source term Q_p are in place. The mass source $\dot{m}'$, the external force $\boldsymbol{f}'$ and the heat source $\dot{\vartheta}'$ as introduced in (36-38) all contribute to the generation of sound. However, it is not the mass flow $\dot{m}$ or the heat flow $\dot{\vartheta}$ which acts as a source, but their time derivatives. Moreover volume flow and heat flow have the same source characteristics, for they appear in the same form. In contrast the forcing $\boldsymbol{f}'$ enters differently. Only the divergence of the specific force $\boldsymbol{f}'/\rho^0$ acts like a source. This is the reason why only the irrotational part of the forcing $\boldsymbol{f}'/\rho^0$ is of acoustic relevance. A rotational part of whatsoever magnitude drops out identically. Note on the other hand, that we assume $\boldsymbol{f}'(\boldsymbol{x}, t)$ (and not $\boldsymbol{f}'/\rho^0$)! to be the independent, prescribed forcing and that even if it were purely rotational the term $\boldsymbol{f}'/\rho^0$ will have an irrotational part unless $\rho^0 = const$. Therefore we may expect an unsteady, purely rotational force to have acoustic significance when it acts in an inhomogeneous domain ($\boldsymbol{\nabla}\rho^0 \neq \boldsymbol{0}$) of the fluid. As an outlook to aeroacoustic source processes we note that generally, unsteady local fluid forces – even when of purely rotational character – may become acoustically significant especially when interacting with inhomogeneities. For instance, we may think of $\boldsymbol{f}'$ as of some suitable model for hydrodynamic forces due to eddies passing through a shock or -less dramatic- eddies subject to strong local acceleration.

Below, in section 2.4.3 we will show that $\sqrt{(a^2)^0}$ represents the speed of sound. Before doing so, it is important to show that apart from the pressure equation (60) wave equations may be alternatively derived for other acoustic quantities as well.

2.4.2.2 Sound field equation for particle velocity. We could also have derived a wave equation for the acoustic particle velocity $\boldsymbol{v}'$. In order to do so, we take the time derivative of the momentum equation (58):

$$\rho^0\frac{\partial^2 \boldsymbol{v}'}{\partial t^2} + \boldsymbol{\nabla}\frac{\partial p'}{\partial t} = \frac{\partial \boldsymbol{f}'}{\partial t}$$

Next, the pressure equation (59) is multiplied by $(a^2)^0$ and the gradient is taken which yields

$$\boldsymbol{\nabla}\frac{\partial p'}{\partial t} + \boldsymbol{\nabla}(\rho^0(a^2)^0\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}') = \boldsymbol{\nabla}\Big((a^2)^0\dot{\theta}'\Big)$$

The pressure may be eliminated from these two equations to obtain

$$\rho^0 \frac{\partial^2 \boldsymbol{v}'}{\partial t^2} - \boldsymbol{\nabla}(\rho^0 (a^2)^0 \boldsymbol{\nabla} \cdot \boldsymbol{v}') = -\boldsymbol{\nabla}((a^2)^0 \dot{\theta}') + \frac{\partial \boldsymbol{f}'}{\partial t}$$

A closer look at this equation for the acoustic particle velocity reveals that i) it is as the wave equation for the pressure (60) a second order equation in time and space, but ii) it has a somewhat different operational form. We can see the differences better when executing the gradient operation on the second term of the above equation $\boldsymbol{\nabla}(\rho^0 (a^2)^0 \boldsymbol{\nabla} \cdot \boldsymbol{v}') = \boldsymbol{\nabla}\cdot \boldsymbol{v}' \, \boldsymbol{\nabla}(\rho^0 (a^2)^0) + \rho^0 (a^2)^0 \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot \boldsymbol{v}')$ (product rule). Further, using the vector identity $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot \boldsymbol{v}') = \Delta \boldsymbol{v}' + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{v}')$ we arrive at

$$\frac{1}{(a^2)^0} \frac{\partial^2 \boldsymbol{v}'}{\partial t^2} - \Delta \boldsymbol{v}' - \frac{\boldsymbol{\nabla} \rho^0 (a^2)^0}{\rho^0 (a^2)^0} \boldsymbol{\nabla}\cdot \boldsymbol{v}' - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{v}') = \frac{1}{\rho^0 (a^2)^0} \Big[-\boldsymbol{\nabla} \Big((a^2)^0 \dot{\theta}' \Big) + \frac{\partial \boldsymbol{f}'}{\partial t} \Big]$$

Notice, that the left hand side of this equation resembles in part the wave equation for the acoustic pressure for constant density ρ^0, eqn (61). However there are two extra terms relating to the spatial variation of the mean density and a term proportional to the vorticity $\boldsymbol{\omega}' = \boldsymbol{\nabla} \times \boldsymbol{v}'$. If we assume $\rho^0 (a^2)^0$ constant, then we see that only the potential part of $\boldsymbol{v}'$ behaves similarly as the sound pressure. We may also say, that if $\boldsymbol{v}'$ is a potential field (this is defined to be the case if it is derivable from a potential function ϕ with $\boldsymbol{v}' = \boldsymbol{\nabla}\phi$), then it behaves similarly as the sound pressure. This shows us, that it is the potential part of the velocity which is related to sound while there may be an additional part (the rotational part $\boldsymbol{v}' - \boldsymbol{\nabla}\phi$) which displays a non-acoustic, namely a vortical behavior. In order to extract the potential part of the velocity one may regard the so called *dilatation* (german: "Dilatation") $\boldsymbol{\nabla}\cdot \boldsymbol{v}'$ as the relevant acoustic quantity. An equation for the dilatation may easily be derived when we divide the second last equation by ρ^0, take the divergence and divide by $(a^2)^0$:

$$\frac{1}{(a^2)^0} \frac{\partial^2 (\boldsymbol{\nabla}\cdot \boldsymbol{v}')}{\partial t^2} - \frac{1}{(a^2)^0} \boldsymbol{\nabla}\cdot \left[\frac{1}{\rho^0} \boldsymbol{\nabla} [\rho^0 (a^2)^0 (\boldsymbol{\nabla}\cdot \boldsymbol{v}')] \right] = Q_v \tag{62}$$

$$Q_v = \frac{1}{(a^2)^0} \boldsymbol{\nabla}\cdot \Big[-\frac{1}{\rho^0} \boldsymbol{\nabla} \Big((a^2)^0 \dot{\theta}' \Big) + \frac{1}{\rho^0} \frac{\partial \boldsymbol{f}'}{\partial t} \Big]$$

For the most common case of a perfect gas this equation can be further simplified. According to (47) we have $\rho^0 (a^2)^0 = \gamma p^0$. Since we assumed that the mean pressure $p^0 = p_\infty$ is constant so is $\rho^0 (a^2)^0$ which we may use in (62) to obtain:

$$\frac{\partial^2 (\boldsymbol{\nabla}\cdot \boldsymbol{v}')}{\partial t^2} - \boldsymbol{\nabla}\cdot \left[(a^2)^0 \boldsymbol{\nabla} (\boldsymbol{\nabla}\cdot \boldsymbol{v}') \right] = \boldsymbol{\nabla}\cdot \Big[-\frac{1}{\rho^0} \boldsymbol{\nabla} \Big((a^2)^0 \dot{\theta}' \Big) + \frac{1}{\rho^0} \frac{\partial \boldsymbol{f}'}{\partial t} \Big]$$

This is the same left hand side as we derived for the dynamics of the acoustic pressure. Therefore the dilatation may be used alternatively to describe the acoustics.

For the most simple case of constant $\rho^0 = \rho_\infty$ and $(a^2)^0 = (a^2)_\infty$ (but not necessarily a perfect gas) equation (62) yields

$$\frac{1}{(a^2)_\infty}\frac{\partial^2(\boldsymbol{\nabla}\cdot\, \boldsymbol{v}')}{\partial t^2} - \Delta(\boldsymbol{\nabla}\cdot\, \boldsymbol{v}') = Q_v \tag{63}$$

$$Q_v = \frac{1}{\rho_\infty}\Big[-\Delta\dot{\theta}' + \frac{1}{(a^2)_\infty}\frac{\partial \boldsymbol{\nabla}\cdot\, \boldsymbol{f}'}{\partial t}\Big]$$

where $\dot{\theta}'$ is as defined in (56).

2.4.2.3 Velocity potential. There is a quite convenient way to describe the acoustic particle velocity and the acoustic pressure perturbation $\boldsymbol{v}'$, p' based on one single function, namely the so called ”velocity potential” ϕ. First, we decompose the velocity perturbation into a purely rotational part $\boldsymbol{v}'_\omega$, which by definition is free of dilatation and another part $\boldsymbol{v}'_d$, which corrects for the dilatation in $\boldsymbol{v}'$:

$$\boldsymbol{v}' =: \underbrace{\boldsymbol{\nabla}\times\boldsymbol{\Psi}}_{:=\, \boldsymbol{v}'_\omega} + \underbrace{\frac{1}{\rho^0}\boldsymbol{\nabla}(\rho^0\,\phi)}_{:=\, \boldsymbol{v}'_d} \tag{64}$$

where $\boldsymbol{\Psi}$ represents an unknown vector stream function which is to be determined along with the unknown velocity potential ϕ. Such a decomposition is also called ”Helmholtz decomposition”; $\boldsymbol{v}'_\omega$ and $\boldsymbol{v}'_d$ are sometimes referred to as ”solenoidal” and ”potential” part of the velocity field. We may introduce a similar decomposition of the (given) disturbance force $\boldsymbol{f}'$:

$$\boldsymbol{f}' =: \underbrace{\rho^0\boldsymbol{\nabla}\times\Big(\frac{1}{\rho^0}\boldsymbol{\Psi}_f\Big)}_{:=\, \boldsymbol{f}'_\omega} + \underbrace{\boldsymbol{\nabla}\phi_f}_{:=\, \boldsymbol{f}'_d} \tag{65}$$

We substitute (64) and (65) into the momentum equation (58) and obtain

$$\boldsymbol{\nabla}\Big(\rho^0\frac{\partial\phi}{\partial t} + p' - \phi_f\Big) = \rho^0\boldsymbol{\nabla}\times\Big(-\frac{\partial\boldsymbol{\Psi}}{\partial t} + \frac{1}{\rho^0}\boldsymbol{\Psi}_f\Big)$$

In order to satisfy this equation we simply set the left bracket and the right bracket separately equal to zero. The first condition provides the relation between the velocity potential and the pressure perturbation:

$$p' = -\rho^0\frac{\partial\phi}{\partial t} + \phi_f \quad . \tag{66}$$

The second condition leaves an equation for the stream function $\boldsymbol{\Psi}$ or respectively an equation for the rotational part of the velocity field:

$$\frac{\partial \boldsymbol{v}'_\omega}{\partial t} = \boldsymbol{\nabla}\times\Big(\frac{1}{\rho^0}\boldsymbol{\Psi}_f\Big) = \frac{1}{\rho^0}\boldsymbol{f}'_\omega \tag{67}$$

Note, that due to (66) the (acoustic) pressure is related to the velocity potential only; the stream function, representing the rotational part of the velocity field is obviously acoustically irrelevant.

Having satisfied the momentum equation we now have to solve the pressure equation (59) as well. Substitution of $\boldsymbol{v}'$ due to (64) and p' due to (66) into (59) yields

$$\frac{1}{(a^2)^0}\frac{\partial}{\partial t}\Big[-\rho^0\frac{\partial\phi}{\partial t} + \phi_f\Big] + \rho^0\boldsymbol{\nabla}\cdot\,\Big[\boldsymbol{\nabla}\times\boldsymbol{\Psi} + \frac{1}{\rho^0}\boldsymbol{\nabla}(\rho^0\,\phi)\Big] = \dot{\theta}'$$

again, $\dot{\theta}'$ is as defined in (56). Taking into account the vector identity $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\ldots=\mathbf{0}$ and writing $\dot{\theta}'$ explicitly this may be re-arranged to

$$\begin{aligned}
\frac{1}{(a^2)^0}\frac{\partial^2\phi}{\partial t^2}-\boldsymbol{\nabla}\cdot\Big(\frac{1}{\rho^0}\boldsymbol{\nabla}(\rho^0\,\phi)\Big) &= Q_\phi \\
Q_\phi &= \underbrace{-\frac{1}{\rho^0}\Big(1-\frac{\sigma^0 p_\infty}{\rho^0 T^0}\Big)\dot{m}'-\frac{\sigma^0}{\rho^0 T^0}\dot{\vartheta}'}+\frac{1}{\rho^0(a^2)^0}\frac{\partial\phi_f}{\partial t} \\
&\quad -\left[\frac{1-\gamma^{-1}}{p_\infty}\dot{\vartheta}'+\frac{1}{\gamma\rho^0}\dot{m}'\right]_{pg}
\end{aligned} \tag{68}$$

We observe that a modified potential $\varphi:=\rho^0\phi$ would obey the same wave operator as the acoustic pressure, see (60):

$$\frac{1}{(a^2)^0}\frac{\partial^2\varphi}{\partial t^2}-\rho^0\boldsymbol{\nabla}\cdot\Big(\frac{1}{\rho^0}\boldsymbol{\nabla}\varphi\Big)=-\Big(1-\frac{\sigma^0 p_\infty}{\rho^0 T^0}\Big)\dot{m}'-\frac{\sigma^0}{T^0}\dot{\vartheta}+\frac{1}{(a^2)^0}\frac{\partial\phi_f}{\partial t}$$

For $\rho^0=const$ the velocity potential ϕ, as well as φ evolve according to the same wave operator as the acoustic pressure p' though due to a different source.

Note that even when $\boldsymbol{\Psi}_f\equiv\mathbf{0}$ (irrotational external force $\boldsymbol{f}'$) the velocity perturbation contains vorticity $\boldsymbol{\omega}'=\boldsymbol{\nabla}\times\boldsymbol{v}'$:

$$\boldsymbol{\omega}'=\boldsymbol{\nabla}\times\left[\frac{1}{\rho^0}\boldsymbol{\nabla}(\rho^0\phi)\right]=-\frac{1}{\rho^{02}}(\underbrace{\boldsymbol{\nabla}\rho^0\times(\boldsymbol{\nabla}\rho^0}_{=\mathbf{0}}\phi+\rho^0\boldsymbol{\nabla}\phi)+\frac{1}{\rho^0}\underbrace{\boldsymbol{\nabla}\times\nabla}_{=\mathbf{0}}(\rho^0\phi)=-\frac{1}{\rho^0}\boldsymbol{\nabla}\rho^0\times\boldsymbol{\nabla}\phi$$

which means, that a "vorticity trace" is generated wherever the gradient of the mean density interacts with the acoustic perturbation (represented by ϕ). For homogeneous Media ($\rho^0=const=\rho_\infty$) the perturbation field is irrotational.

The wave equation for the velocity potential ϕ allows to look at several asymptotic cases which give an insight into the physics of an acoustic problem. Let us take the simplest case of (68), i.e., a homogeneous density $\rho^0=const=\rho_\infty$. Then we have

$$\underbrace{\frac{1}{(a^2)_\infty}\frac{\partial^2\phi}{\partial t^2}-\Delta\phi}_{=:\,L[\phi]}=Q_\phi$$

Let us look at this equation when there are no sources present, i.e. $Q_\phi=0$. Now consider three asymptotic cases

$\lim_{(a^2)_\infty\to\infty} L[\phi] = -\Delta\phi$ — incompressible fluid (remember (42); $1/a^2=\partial\rho/\partial p$)

$\lim_{\partial^2/\partial t^2\to 0} L[\phi] = -\Delta\phi$ — weak unsteadiness, i.e. slow processes

$\lim_{r\to 0} L[\phi] = -\Delta\phi$ — behavior of ϕ near a singularity at $r=0$ (r - distance to singularity). Here, ϕ locally behaves like $\phi\simeq r^{-n}$ for some positive number n. This implies, that near $r=0$ in $L[\phi]$ the part $\frac{\partial^2\phi}{\partial t^2}\simeq r^{-n}$, while $|\Delta\phi|\simeq|\frac{\partial^2\phi}{\partial r^2}|\simeq r^{-n-2}\gg|\frac{\partial^2\phi}{\partial t^2}|$.

In all of the three cases the wave operator $L[\ldots]$ reduces to the Laplace operator acting on the velocity potential $\Delta\phi$. But we know that $\Delta\phi = 0$ describes incompressible potential flow. Therefore we see that the acoustic particle velocity $\boldsymbol{v}' = \boldsymbol{\nabla}\phi$ in the immediate neighborhood of a singularity like a point source/sink or a geometry edge behaves like an incompressible potential flow. Note however, that this is not necessarily the case for the pressure field. Given the velocity field for an incompressible potential flow problem one would determine the pressure according to Bernoulli's equation (proportional to the square of the velocity). This corresponds to the acoustic pressure field due to (66) only in the limit of asymptotically slow flow speeds. The above relations also say that very slowly changing sound velocity fields again behave like an incompressible fluid.

2.4.2.4 Helmholtz equation. In acoustics, transient (starting) processes are of rare practical importance. Therefore the wave equation is often conveniently expressed in the frequency domain ω. The transition from time to frequency of a function $h'(\boldsymbol{x}, t)$ and vice versa is defined through the Fourier transform and its inverse, eqns(23, 24). Fourier transforming the wave equation (61) we obtain the so-called "reduced wave equation" or "Helmholtz equation"

$$-k^2\hat{p} - \Delta\hat{p} = \hat{Q}_p \quad , \tag{69}$$

in which $\hat{p}(\omega, \boldsymbol{x})$ is the Fourier coefficient of $p'(t, \boldsymbol{x})$ for the circular frequency ω. We have grouped the frequency ω and $(a^2)_\infty$ together to the so called wave number

$$k := \frac{\omega}{\sqrt{(a^2)_\infty}}. \tag{70}$$

2.4.3 Plane-, spherical- and cylindrical free acoustic waves

In this section we consider the case of no sources present, i.e. we are not interested in how perturbations p' originated, but simply assume they are existing and we study their dynamics due to the homogeneous wave equation, i.e. (61) with $Q_p \equiv 0$. Moreover, let us for the moment look at small perturbations to a stagnant fluid of initially constant density and pressure throughout the unbounded space.

2.4.3.1 Plane- or one dimensional waves d'Alembert gave a very general solution to the homogeneous wave equation (61) for homogeneous medium in one dimension (say x):

$$p'(x, t) = f(t - x/a_\infty) + g(t + x/a_\infty) \tag{71}$$

where f and g are arbitrary functions and with the abbreviation $a_\infty = \sqrt{(a^2)_\infty}$. By inserting (71) into (61) one immediately shows that the wave equation is indeed satisfied. If the independent space variable is not x but more generally $x_k := \boldsymbol{e}_k{\cdot}\boldsymbol{x}$ with $\boldsymbol{e}_k$ an arbitrary unit vector one may

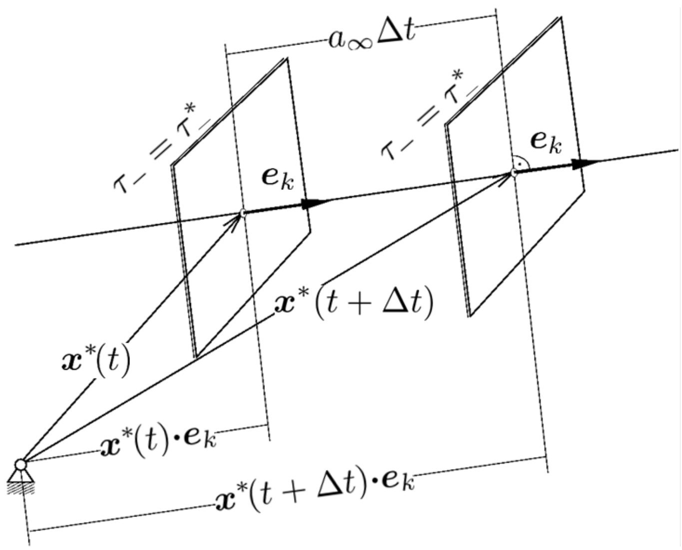

Figure 10: Plane wave propagation of signal $f(\tau_-)$.

write (71) as

$$p'(\boldsymbol{x}, t) = f(\underbrace{t - \boldsymbol{x} \cdot \boldsymbol{e}_k / a_\infty}_{=:\,\tau_-}) + g(\underbrace{t + \boldsymbol{x} \cdot \boldsymbol{e}_k / a_\infty}_{=:\,\tau_+}) \tag{72}$$

In order to interpret d'Alembert's solution (72) physically it is useful to first consider a fixed function value of f, i.e.

$$f^* = f(\underbrace{\tau_-^*}) $$
$$\tau_-^* = t - \boldsymbol{x}^* \cdot \boldsymbol{e}_k / a_\infty \Longrightarrow x_k^* = \boldsymbol{x}^* \cdot \boldsymbol{e}_k = a_\infty t - a_\infty \tau_-^*$$

The last expression is the equation for a plane in space with a normal vector $\boldsymbol{e}_k$ "Hessesche Normalform", i.e. the function values of f are constant on such a plane. The relation shows as well that as time goes on, these planes are shifted along their normal direction of $\boldsymbol{e}_k$ with the speed a_∞. A fixed function value of f is also called "signal". Since all possible function values of f move with the same speed a_∞ the shape of $f(\tau_-)$ does not change (see also left of figure *11*). Since all signals are constant on planes, the solution is also called *plane wave* (german: "ebene Welle"). The situation is again shown in figure *10*.

The interpretation of the g-part of d'Alembert's solution (72) is analogous, except that here $x_k^* = \boldsymbol{x}^* \cdot \boldsymbol{e}_k = -a_\infty t + a_\infty \tau_+$. That means, that g describes a signal, which moves with the speed a_∞ in the negative direction of $\boldsymbol{e}_k$.

Now we know how to interpret the meaning of the thermodynamic variable a^2 defined in (42) and evaluated for a perfect gas in (47): it is the square of the speed at which small pressure perturbations travel through a homogeneous medium at rest, i.e. the *speed of sound* (german: "Schallgeschwindigkeit").

$$a_0 = \sqrt{(a^2)^0} \tag{73}$$

For air as a perfect gas with $\gamma = 1.4$ and a specific gas constant of $R = 287 J/kgK$ at $T^0 = 293K$

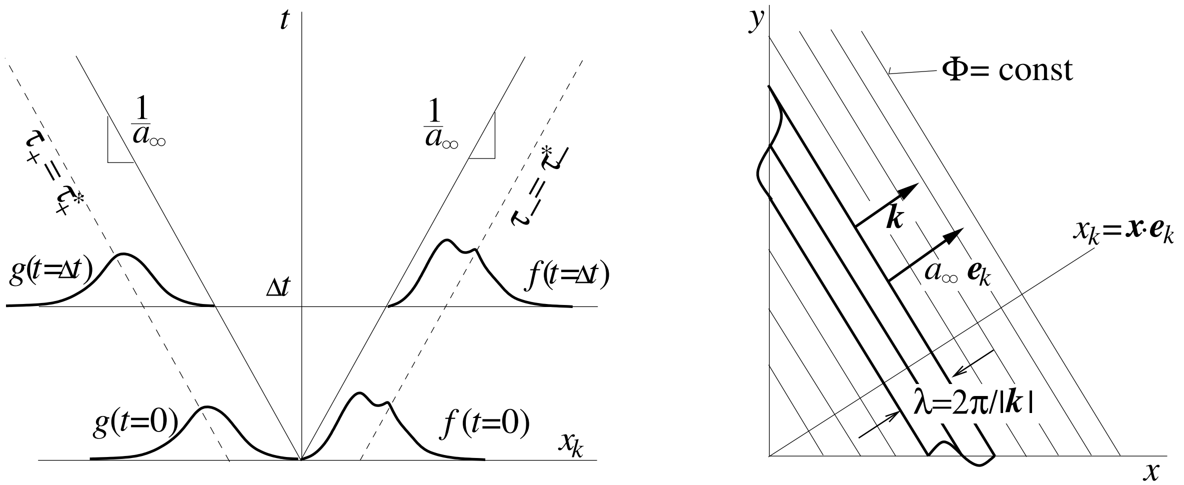

Figure 11: Plane waves. Left: example of general solution due to d'Alembert, right: periodic wave as solution of Helmholtz equation

we obtain for instance $a_0 = \sqrt{\gamma R T^0} = 343 m/s$.

We could have solved our homogeneous wave equation (61) also on the basis of a separation of variables like $p' = C\exp(i\omega t)\exp(-ik_x x)\exp(-ik_y y)\exp(-ik_z z) = C\exp(i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x})$, where $\boldsymbol{k} = (k_x, k_y, k_z) = k\boldsymbol{e}_k$ with $k := |\boldsymbol{k}|$ is called the wave number vector and ω is called the circular frequency all being some constant numbers. If we write this as $p' = C\exp(i\Phi)$

$$\Phi = \omega t - \boldsymbol{k}\cdot\boldsymbol{x} \tag{74}$$

is called the *wave phase* or *phase function*, german: "Wellenphase" or "Phasenfunktion" (see right of figure *11*). Upon substitution into the homogeneous wave equation (61) we obtain the determining equation for $\boldsymbol{k}$, called the *dispersion relation* (german: "Dispersionsrelation"):

$$|\boldsymbol{k}|^2 = \omega^2/(a^2)_\infty = \omega^2/a_\infty^2 \tag{75}$$

for a given frequency ω there are obviously two solutions for k, namely $k = \pm\omega/a_\infty$, yielding the following expression for the pressure:

$$p' = C_1\exp[i\omega(t - \boldsymbol{x}\cdot\boldsymbol{e}_k/a_\infty)] + C_2\exp[i\omega(t + \boldsymbol{x}\cdot\boldsymbol{e}_k/a_\infty)]$$

Obviously, this time-harmonic solution (pure tone) is a special case of d'Alembert's general solution, whereby $f(\tau_-) = C_1\exp(i\omega\tau_-)$ and $g(\tau_+) = C_2\exp(i\omega\tau_+)$.

Note that the time derivative of the phase is the circular frequency ω, whereas the magnitude of its spatial gradient represents the wave number k:

$$\frac{\partial\Phi}{\partial t} = \omega =: 2\pi/T \tag{76}$$

$$|\boldsymbol{\nabla}\Phi| = k =: 2\pi/\lambda \tag{77}$$

T is called *period* (german:"Schwingungsdauer") and λ is called *wavelength* (german: "Wellenlänge"). Let us look at surfaces of constant phase $\Phi(\boldsymbol{x}, t) = const$, i.e. those, where the solution p' remains constant. These surfaces are called *wave fronts* (german: "Wellenfronten"). We may think of plane waves as waves originating from a source which is homogeneously distributend in a plane (two-dimensional source).

2.4.3.2 Spherical- or threedimensional waves Let us now look at solutions to the wave equation in spherical co-ordinates, defined through the cartesian co-ordinates x, y, z by $x = r\sin\vartheta\cos\varphi$, $y = r\sin\vartheta\sin\varphi$, $z = r\cos\vartheta$. In these co-ordinates, the wave equation (61) reads

$$\frac{1}{a_\infty^2}\frac{\partial^2 p'}{\partial t^2} - \left[\frac{\partial^2 p'}{\partial r^2} + \frac{2}{r}\frac{\partial p'}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin^2\vartheta}\frac{\partial^2 p'}{\partial\varphi^2} + \frac{\partial^2 p'}{\partial\vartheta^2} + \frac{1}{\tan\vartheta}\frac{\partial p'}{\partial\vartheta}\right)\right] = 0 \quad (78)$$

We are interested in the simple case of a solution symmetric with respect to the point $r = 0$ for which all the terms, involving φ and ϑ in (78) vanish. Upon grouping p' and r into a new variable $p'r$ we may write the wave equation like

$$\frac{1}{a_\infty^2}\frac{\partial^2 p'r}{\partial t^2} - \frac{\partial^2 p'r}{\partial r^2} = 0 \quad (79)$$

which shows that the solution for the spherical wave $p'(r,t)$ can be obtained from the one-dimensional solution for the variable $p'r$. Only the point $r = 0$ is somewhat critical, since p' becomes indefinite. We immediately use d'Alembert's solution (72) and we find

$$p'(r,t) = \frac{1}{r}\left[f(t - r/a_\infty) + g(t + r/a_\infty)\right] \quad (80)$$

As for plane waves the shape of the wave does not change, it only suffers from a decrease in magnitude like the inverse of the distance r. The function f represents a wave spreading away from the center $r = 0$ (so called outgoing) while g is running towards, i.e. collapsing into the center (so called incoming). The distinction between these two parts of the solution is rather important as far as the role of cause and effect is concerned. This shall be shortly discussed next.

Consider the process of switching on a source in the domain $r < R$, "source" being understood as to representing the cause of an acoustic pressure signal. Before this moment the field is assumed to be free of any signals, which translates into the fact that before a certain observation time instant, which we denote $t = 0$, no pressure field can be observed: $p'(r > R, t < 0) = 0$. Translated to d'Alembert's solution (80) this may be expressed like

$$0 = f(\underbrace{t - r/a_\infty}_{\tau_-}) + g(\underbrace{t + r/a_\infty}_{\tau_+}), \quad \text{for } t < 0 \text{ and } r > R$$

Now, f and g are independent functions and therefore cannot cancel each other for $t < 0$, which implies, that $f(t < 0) = 0$ and $g(t < 0)$ individually vanish. Let us first look at what this means for f. On the one hand we know that the function values of f remain constant on lines $r^* = a_\infty t - a_\infty\tau_-^*$ any time, unless the function value is modified along this line due to the action of a source. This is not the case for all lines, for which $r^*(t = 0) > R$. The border line at the outer edge of the source domain would be $r^*(t) = a_\infty t + R$ and thus we know, that in the domain $r^* > a_\infty t + R$ there can never be a non-zero signal originating from the f-part of d'Alembert's solution. This is illustrated in the left of figure *12*. Analogous reasoning leads to the result, that the incoming part of the solution g can exist only for a finite period of time after the source is switched on (see right of figure *12*).

One more remark here: physically, the "g part" of d'Alembert's solution still leaves a trace in the field also at any time $t > R/a_\infty$. The reason results again from physics rather than mathematics, namely that the radial component of the acoustic particle velocity, caused by the g-signal

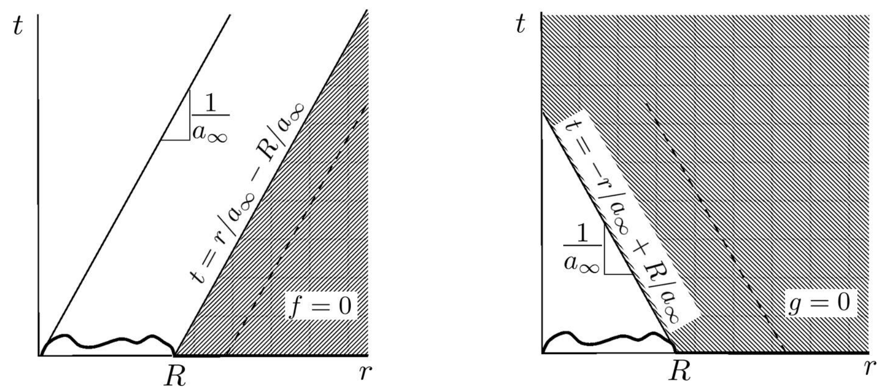

Figure 12: Left: solution domains for outgoing part of solution f, right: solution domains for incoming part of solution g.

collapsing into the origin at $r = 0$, must vanish at any time. This condition can only be satisfied by an additional "f" part, which is generated at the origin and appears as the inverted, reflected $g-$signal, traveling in positve direction r, see also example in appendix F.1.

The mentioned restrictions on f and g make sure that the signal p' can originate only from the source domain, but not from anywhere else. This is called *causality* condition. A violation of causality would make the mathematical solution unphysical. This implies the assumption that in open domains, no waves are coming from infinity (radiation condition). Note that for the special (and practically very important) case of a point source, i.e. a source domain with $R \to 0$ the incoming part of the solution vanishes completely: $g(\tau_+, R \to 0) \equiv 0$. Then $p' = f(\tau_-)/r$ from (80) is the most general causal solution to the wave equation. The solution f automatically fulfills the radiation condition. The satisfaction of this radiation condition has to be required explicitly, when we compute the pressure in the frequency domain by solving the Helmholtz equation. Then we would not have arrived at (80) but rather similarly at

$$\hat{p}(r, k) = \frac{1}{r}\left[\hat{f}\exp(-ikr) + \hat{g}\exp(ikr)\right] \tag{81}$$

and in conjunction with the common time factor $\exp(+i\omega t)$ going together with $\hat{p}$ as mentioned in (70) only the first term represents waves running towards increasing r. As above we sort out the $\hat{g}$-part of (81) in order to meet the radiation condition. Note that we would have arrived at the same solution had we Fourier transformed (80).

We may think of spherical waves as waves originating from a concentrated point source (zero-dimensional source).

The radiation condition can be formulated for general cases, not requiring spherically symmetric solutions $p' = p'(r)$ but $p' = p'(r, \varphi, \vartheta)$. Let us re-consider the wave equation in spherical polar co-ordinates

$$\frac{1}{a_\infty^2}\frac{\partial^2 p'r}{\partial t^2} - \frac{\partial^2 p'r}{\partial r^2} = \frac{1}{r^2}\left(\frac{1}{\sin^2\vartheta}\frac{\partial^2 p'r}{\partial\varphi^2} + \frac{\partial^2 p'r}{\partial\vartheta^2} + \frac{1}{\tan\vartheta}\frac{\partial p'r}{\partial\vartheta}\right)$$

We observe, that for very large distances from the origin $r \to \infty$ the r.h.s. vanishes and although p' will even here be a function of the orientations φ and ϑ, the equation tells us, that for large r the dynamics of p' is governed by r only. Therefore the solution for very large r will again be (80) with the generalization, that f and g now are functions of the orientations, too. D'Alembert's solution (80) enabled us to conveniently sort out the waves running towards decreasing r (the waves $g(t + r/a_\infty)$ coming from infinity). The pressure then is $p'r = f(t - r/a_\infty, \varphi, \vartheta)$.

If we eliminate f by differentiating w.r.t. r and t, respectively and transform the result to the frequency domain we obtain the "Sommerfeld radiation condition"

$$\lim_{r\to\infty} r\left(ik\hat{p} + \frac{\partial \hat{p}}{\partial r}\right) = 0 \tag{82}$$

Just as we have to respect causality when solving the wave equation for $p'(\boldsymbol{x}, t)$, we must make sure that the radiation condition is satisfied when solving the Helmholtz equation for $\hat{p}(\boldsymbol{x}, \omega)$.

Let us again look at the sound field at large distances from the wave centers, i.e. $r \to \infty$. Since the velocity potential ϕ is governed by the same wave operator as the pressure, d'Alembert's solution applies also for ϕ. Moreover, for large distances just as above we find that in the general case $\phi r = f(t - r/a_\infty, \varphi, \vartheta)$. We may therefore express p' from (66) without sources like $p' = -\rho_\infty \frac{\partial \phi}{\partial t} = -\frac{\rho_\infty}{r}\frac{df}{d\tau_-}$ and $v'_r = \frac{\partial \phi}{\partial r}$ with constant ρ_∞ due to (64). For large r this gives $v'_r = -\frac{1}{a_\infty r}\frac{df}{d\tau_-}$. One may therefore eliminate f using p', which yields

$$p'(r \to \infty) = \rho_\infty a_\infty \; v'_r(r \to \infty). \tag{83}$$

in the farfield the pressure and acoustic particle velocity are proportional. In analogy to Ohm's law from electricity $U = RI$ with the correspondences $p' = U$ and $v'_r = I$ the expression $z_\infty := \rho_\infty a_\infty$ is called *wave drag* or *free field impedance* (german: "Wellenwiderstand" or "Freifeldimpedanz"). Note that with (83) one may express the intensity level in terms of the pressure level in the farfield, because $I = \overline{v'_r p'} \simeq \overline{p'^2}/(\rho_\infty a_\infty)$. We finally obtain

$$L_I \simeq L_p - 10\lg\left(\frac{z_\infty}{z_{\text{ref}}}\right) \quad \text{with } z_{\text{ref}} = \frac{p^2_{\text{ref}}}{I_{\text{ref}}} = 4 \cdot 10^2 \frac{kg}{m^2 s} \quad \text{and } z_\infty = \rho_\infty a_\infty \tag{84}$$

2.4.3.3 Cylindrical- or two-dimensional waves The solution of the wave equation in two dimensions appears to be more complicated as in one and three dimensions. This is seen when writing the wave equation in axi-symmetric co-ordinates (r, θ, z), defined through the cartesian co-ordinates by $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. In these co-ordinates, the wave equation (61) without sources reads

$$\frac{1}{a^2_\infty}\frac{\partial^2 p'}{\partial t^2} - \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p'}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p'}{\partial \theta^2} + \frac{\partial^2 p'}{\partial z^2}\right] = 0 \tag{85}$$

Again, looking at axi-symmetric solutions we have

$$\frac{1}{a^2_\infty}\frac{\partial^2 p'}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p'}{\partial r}\right) = 0 \tag{86}$$

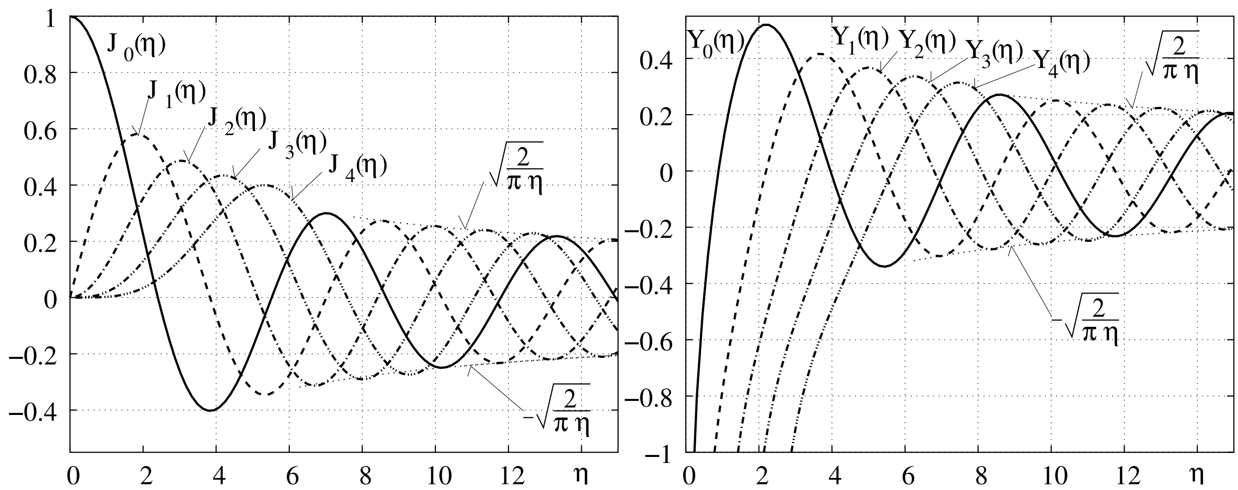

Figure 13: Bessel functions of first kind (left) and second kind (right).

Unfortunately there is no exact way to reduce this equation to one for a plane wave, which already shows, that the wave form is not conserved as in one or three dimensions. Cylindrical waves typically show a trailing wake. The equation has to be solved directly. We are separating out the time like in (70) and now rather solve the Helmholtz equation (69) in axi-symmetric co-ordinates:

$$\frac{\partial^2 \hat{p}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \hat{p}}{\partial \eta} + \hat{p} = 0 \tag{87}$$

where we have non-dimensionalized r like $\eta = kr$. This is Bessel's differential equation. An elementary solution is not available, but since the differential equation is linear and of second order all solutions can be expressed in terms of two linearly independent universal solutions, which are tabulated, namely the so called Bessel function of first kind, zeroth order J_0 and the Bessel function of second kind, zeroth order Y_0 (also Weber's function), see figure *13*.

$$\hat{p} = AJ_0(\eta) + BY_0(\eta) \tag{88}$$

where A and B are constants. Again, we are only interested in outgoing solutions, i.e. those, which run towards increasing r or η respectively. For the assumed time factor $\exp(i\omega t)$ the only linear combination for which this is possible is $A = P$ and $B = -iP$. Therefore the cylindrical wave solution is expressed as

$$\hat{p} = PH_0^{(2)}(\eta) = P[(J_0(\eta) - iY_0(\eta))] \tag{89}$$

which defines the so called Hankel function of second kind and zeroth order $H_0^{(2)}$ (the one of the first kind $H_0^{(1)}$ would in combination with $\exp(i\omega t)$ render incoming waves). It is interesting to see, how the solution behaves close to the origin $\eta \to 0$ and for very large relative distances $\eta \to \infty$:

$$H_0^{(2)}(\eta) \simeq -\frac{2i}{\pi}\ln(\eta) \qquad \text{for} \quad \eta \to 0 \tag{90}$$

$$H_0^{(2)}(\eta) \simeq \sqrt{\frac{2}{\pi\eta}}\exp[-i(\eta - \pi/4)] \qquad \text{for} \quad \eta \to \infty \tag{91}$$

Therefore for large distances the cylindrical wave again behaves similar to a plane wave (the wave form does not change asymptotically anymore) with the signal decreasing like $1/\sqrt{r}$. We may think of cylindrical waves as waves originating from a source which is homogeneously distributed along a straight line (one-dimensional source).

The Sommerfeld radiation condition for axi-symmetric waves translates to

$$\lim_{r\to\infty} \sqrt{r}\Big(ik\hat{p} + \frac{\partial \hat{p}}{\partial r}\Big) = 0 \tag{92}$$

2.4.4 More general elementary solutions to the homogenous wave equation

New elementary solutions to the homogeneous wave equation for constant density ρ_∞ may be derived from the ones determined in the previous section. This is due to the fact that the governing equation is linear with constant coefficients. If we take e.g. the derivative along a cartesian direction, say $\tilde{x}$, of the homogeneous form of the wave equation (61), scaled by some constant c, then we get

$$\frac{\partial}{\partial \tilde{x}}\left\{\frac{1}{a_\infty^2}\frac{\partial^2 cp'}{\partial t^2} - \Delta(cp')\right\} = \frac{1}{a_\infty^2}\frac{\partial^2 p_2'}{\partial t^2} - \Delta p_2' = 0 \quad \text{with } p_2' := c\frac{\partial p'}{\partial \tilde{x}} \tag{93}$$

This result clearly shows that for constant fluid density ρ_∞ and speed of sound a_∞ the (spatial and temporal) derivatives of solutions to the wave equation are again solutions to the wave equation.

2.4.5 From a source to its sound field in free space

In the previous section we have investigated some elementary solutions to the wave- and Helmholtz equation. In this way we were able to interpret the introduced thermodynamic quantity $\sqrt{(a^2)^0}$ as the speed of sound, i.e. the speed at which the phase of a small amplitude acoustic wave travels.

Here we want to look at more general solutions and trace them back to their origin. In this way we will find the relation between the source term Q_p of (61) and the solution p'. In this section we restrict ourselves to the free field (no obstacles present).

Let us re-examine our spherically symmetric causal solution from (80), centered not at the origin of the system $\boldsymbol{x}$ with $|\boldsymbol{x}| = r$, but at a general position $\boldsymbol{\xi}$ (see figure *14*). The general form of the solution may not change, i.e. the pressure is to depend exclusively on the distance

$$r := |\boldsymbol{x} - \boldsymbol{\xi}| \tag{94}$$

from the field's center. With this shift in origin we have from (80)

$$p_\xi'(\boldsymbol{x}, t, \boldsymbol{\xi}) = \frac{1}{r} f(\tau, \boldsymbol{\xi}) \tag{95}$$

$$\tau = t - r/a_\infty \tag{96}$$

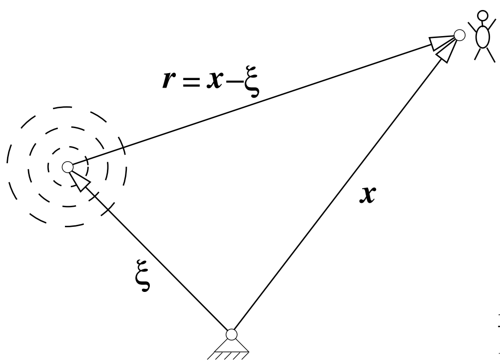

Figure 14: Spherically symmetric sound field described for symmetry point shifted out of origin by $\boldsymbol{\xi}$.

Here we have introduced the variable τ, which is called "retarded time", because it denotes a moment in the past. Note that our solution satisfies the homogeneous wave equation everywhere, but at $\boldsymbol{x} = \boldsymbol{\xi}$, i.e. at $r = 0$, where it is not determined. Except on this point our solution describes waves running away from $\boldsymbol{x} = \boldsymbol{\xi}$, which may let us anticipate, that they are excited here, and that consequently $\boldsymbol{\xi}$ is a source point. In fact, the domain, where the homogeneous wave equation is violated (i.e. the r.h.s. of (60) is non-zero) is called source domain.

In order to establish the connection between the acoustic field and its source, let us first construct a more general pressure field solution of the homogeneous wave equation. We will then extend this solution into the source region and obtain the desired relation. A more general solution of the homogeneous wave equation may be obtained by superposition of several solutions (95), each centered at different $\boldsymbol{\xi}$. In the limit of infinitely many, continuously arranged fields superposition means integration $p'(\boldsymbol{x}, t) = \int\limits_{V_S} p'_\xi(\boldsymbol{x}, t; \boldsymbol{\xi}) dV(\boldsymbol{\xi})$, i.e. we assume p' to be composed like

$$p'(\boldsymbol{x}, t) = \int\limits_{V_S} \frac{f(\tau(t, r); \boldsymbol{\xi})}{r} dV(\boldsymbol{\xi}) \tag{97}$$

The integration volume V_S is the domain, defined by the locations of the centres $\boldsymbol{\xi}$ of all partial fields the overall field p' is composed of. For physical realizability reasons let us assume that V_S is finite. Note that we may expect p' to be causal since it is strictly made up of causal components. First we need to check whether indeed (97) satisfies the homogeneous wave equation (61)

$$\frac{1}{a_\infty^2} \frac{\partial^2 p'}{\partial t^2} - \Delta_x p' = 0$$

where we underline with the notation Δ_x, that the differentiations in Δ are to be carried out at the point $\boldsymbol{x}$ because according to (97) we have the pressure as $p'(\boldsymbol{x}, t)$. This means that Δ_x can be taken inside the integral of p', because the integration variable is $\boldsymbol{\xi} \neq \boldsymbol{\xi}(\boldsymbol{x})$:

$$\frac{1}{a_\infty^2} \frac{\partial^2 p'}{\partial t^2} - \Delta_x p' = \int\limits_{V_S} \left\{ \frac{1}{a_\infty^2 r} \frac{\partial^2 f(\tau, \boldsymbol{\xi})}{\partial t^2} - \Delta_x \Big(\frac{f(\tau, \boldsymbol{\xi})}{r} \Big) \right\} dV(\boldsymbol{\xi}) \stackrel{?}{=} 0 \tag{98}$$

Let us look at the second term and re-formulate $\Delta_x(f/r) \equiv \boldsymbol{\nabla}_x \boldsymbol{\cdot} \boldsymbol{\nabla}_x(f/r) = (1/r)\Delta_x f + 2(\boldsymbol{\nabla}_x f) \boldsymbol{\cdot} \boldsymbol{\nabla}_x(1/r) + f \Delta_x(1/r)$. The essential step is to express $\boldsymbol{\nabla}_x f$ in terms of the time derivative. We have $\boldsymbol{\nabla}_x f = \frac{\partial f}{\partial \tau} \boldsymbol{\nabla}_x \tau$, and further $\Delta_x f = \boldsymbol{\nabla}_x \boldsymbol{\cdot} (\frac{\partial f}{\partial \tau} \boldsymbol{\nabla}_x \tau) = \frac{\partial^2 f}{\partial \tau^2} (\boldsymbol{\nabla}_x \tau)^2 + \frac{\partial f}{\partial \tau} \Delta_x \tau$. Now according to (96) $\boldsymbol{\nabla}_x \tau = -(1/a_\infty)\boldsymbol{\nabla}_x r$, which we use to finally obtain

$$\Delta_x\Big(\frac{f(\tau,\boldsymbol{\xi})}{r}\Big) = \overbrace{\frac{1}{a_\infty^2 r}\frac{\partial^2 f}{\partial \tau^2}\underbrace{(\boldsymbol{\nabla}_x r)^2}_{=1} - \frac{1}{a_\infty r}\frac{\partial f}{\partial \tau}\underbrace{\Delta_x r}_{=2/r}}^{(1/r)\Delta_x f} + \overbrace{\frac{2}{a_\infty r^2}\frac{\partial f}{\partial \tau}\underbrace{(\boldsymbol{\nabla}_x r)^2}_{=1}}^{2(\boldsymbol{\nabla}_x f)\cdot\boldsymbol{\nabla}_x(1/r)} + f\Delta_x(1/r)$$

Where we have indicated that $\boldsymbol{\nabla}_x r =: \boldsymbol{e}_r$ is nothing but the unit vector pointing along the vector $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{\xi}$, which is seen by elementary differentiation on r defined by (94). In the same way one verifies that $\Delta_x r = 2/r$. Note that the inner two terms in the above relation cancel each other. Inserting our expression back into (98) gives

$$\frac{1}{a_\infty^2}\frac{\partial^2 p'}{\partial t^2} - \Delta_x p' = -\int\limits_{V_S} f(\tau,\boldsymbol{\xi})\Delta_x(1/r)\, dV(\boldsymbol{\xi}) \stackrel{?}{=} 0 \tag{99}$$

where we have used $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial \tau}$. We have to check, under which conditions the above integral is indeed zero. Otherwise the homogeneous wave equation is not satisfied. Again, the only peculiarity we expect is certainly at postitions where $r = 0$. For all other points we have $\Delta_x(1/r) = \boldsymbol{\nabla}_x\cdot\boldsymbol{\nabla}_x(1/r) = -\boldsymbol{\nabla}_x\cdot(r^{-2}\boldsymbol{\nabla}_x r) = r^{-2}\Delta_x r - 2r^{-3}(\boldsymbol{\nabla}_x r)^2 = 0$. Thus we have shown, that p' due to (97) satisfies the wave equation except at all points where $r = 0$. Note, that this condition is satisfied, for all points $\boldsymbol{x} \notin V_S$.

The points $\boldsymbol{x} \in V_S$ remain to be considered, because in this case $r = 0$ is met, namely at $\boldsymbol{x} = \boldsymbol{\xi}$. Let us first note, that for a given observer position $\boldsymbol{x}$ we may shrink the integration volume V_S equivalently to an infinitesimal domain V_ϵ, say a sphere of radius $\epsilon \to 0$, because we have shown the integrand to be zero outside V_ϵ. We assume $f(\tau = t - \epsilon/a_\infty, \boldsymbol{\xi} = \boldsymbol{x} - \epsilon\boldsymbol{e}_r)$ to be a continuous function of space. Therefore, as $\epsilon \to 0$ we have $f(\tau,\boldsymbol{\xi}) \to f(t,\boldsymbol{x})$ and we may take it out of the integral (99). Then we note again that since r directly couples $\boldsymbol{\xi}$ to $\boldsymbol{x}$ we may write $\boldsymbol{\nabla}_x r = -\boldsymbol{\nabla}_\xi r$ and $\boldsymbol{\nabla}_x\cdot\boldsymbol{g}(r) = -\boldsymbol{\nabla}_\xi\cdot\boldsymbol{g}(r)$ for any function $\boldsymbol{g}$, such that we can express $\Delta_x(1/r) = \boldsymbol{\nabla}_x\cdot(-r^{-2}\boldsymbol{\nabla}_x r)$ to obtain

$$\int\limits_{V_\epsilon} f(\tau,\boldsymbol{\xi})\Delta_x(1/r)\, dV(\boldsymbol{\xi}) = f(t,\boldsymbol{x})\int\limits_{V_\epsilon} \boldsymbol{\nabla}_\xi\cdot(-r^{-2}\boldsymbol{\nabla}_\xi r)\, dV(\boldsymbol{\xi}) = -f(t,\boldsymbol{x})\int\limits_{\partial V_\epsilon} \underbrace{(-e_r\cdot\boldsymbol{n})\frac{dS(\boldsymbol{\xi})}{r^2}}_{=:\, d\Omega}$$

where for the last expression we have used Gauss' theorem to convert the integral over the sphere's volume to one over its surface, whose outward pointing local normal is denoted $\boldsymbol{n}$. We have also substituted $\boldsymbol{\nabla} r = -\boldsymbol{e}_r$ the unit vector pointing radially away from the center point $\boldsymbol{x}$. Note, that the last term in the integral identity is nothing but the definition of the space angle $\Omega(\boldsymbol{x})$ of the surface ∂V_ϵ about the point $\boldsymbol{x}$. Since the surface is closed and $\boldsymbol{x}$ inside of it, our integral equals 4π. Re-inserting this into (99) we have for all points $\boldsymbol{x} \in V_S$

$$\frac{1}{a_\infty^2}\frac{\partial^2 p'}{\partial t^2} - \Delta_x p' = 4\pi f(t,\boldsymbol{x}) \neq 0 \tag{100}$$

Let us compare this with (61), the inhomogeneous wave equation. We immediately see that the source term $Q_p = 4\pi f(t,\boldsymbol{x})$. Note also that there is no restriction on Q_p due to f, because according to d'Alembert f is an arbitrary function. Therefore we have found the general solution of (61) with no boundaries present to be

$$p'(\boldsymbol{x},t) = \frac{1}{4\pi}\int\limits_{V_S} \frac{Q_p(\tau(t,r),\boldsymbol{\xi})}{r} dV(\boldsymbol{\xi}) \tag{101}$$

where again for reference $\tau(t,r) = t - r/a_\infty$ and $r = |\boldsymbol{x} - \boldsymbol{\xi}|$. We may express this solution as well in the frequency domain (23). The Fourier integral $\int Q_p(t - r/a_\infty)\exp(-i\omega t)dt$ of the source at the retarded time $\tau = t - r/a_\infty$ is computed by changing the integration variable from t to τ, i.e. $\int Q_p(\tau)\exp(-i\omega\tau)d\tau \exp(-ikr)$ with $k = \omega/a_\infty$ the wave number. The time retardation occurs as space factor $\exp(-ikr)$. Now the free space solution (101) reads

$$\hat{p}(\boldsymbol{x},\omega) = \frac{1}{4\pi}\int\limits_{V_S} \frac{\hat{Q}_p(\omega,\boldsymbol{\xi})\exp(-ikr)}{r} dV(\boldsymbol{\xi}) \tag{102}$$

The general formulae (101, 102) do not tell us much about the character of the pressure field resulting from Q_p. At first glance it may seem that we essentially would have to expect some sort of $\frac{1}{r}$-dependence, if r measures a characteristic distance from the source. This however is usually true only for very large r compared with a characteristic diameter of the source volume V_S. Near the source region V_S this dependence may change in character. In order to obtain a slightly better insight into what sort of acoustic field we may expect from Q_p we examine what Q_p is actually composed of. We take its definition in the case $a_0 = const = a_\infty$ from (61)

$$Q_p(t,\boldsymbol{\xi}) = \frac{\partial}{\partial t}\Big\{\Big(1 - \frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty}\Big)\dot{m}'(t,\boldsymbol{\xi}) + \frac{\sigma_\infty}{T_\infty}\dot{\vartheta}'(t,\boldsymbol{\xi})\Big\} - \boldsymbol{\nabla}_\xi\cdot\boldsymbol{f}'(t,\boldsymbol{\xi})$$

The essential step, when inserting $Q_p(t,\boldsymbol{\xi})$ into the solution formula (101) is its evaluation at the retarded time $\tau(t,r)$. So far we have assumed non-moving sources and therefore r_ξ is time independent. The terms due to the externally forced mass flux density $\dot{m}'$ and heat flux density $\dot{\vartheta}'$ are easy to evaluate at the retarded time, because for our non-moving sources $\frac{\partial}{\partial t} = \frac{\partial}{\partial\tau}$ such that here we may simply replace t by τ, i.e.

$$\left[\frac{\partial}{\partial t}\Big\{\Big(1 - \frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty}\Big)\dot{m}'(t,\boldsymbol{\xi}) + \frac{\sigma_\infty}{T_\infty}\dot{\vartheta}'(t,\boldsymbol{\xi})\Big\}\right]_\tau = \frac{\partial}{\partial\tau}\Big\{\Big(1 - \frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty}\Big)\dot{m}'(\tau,\boldsymbol{\xi}) + \frac{\sigma_\infty}{T_\infty}\dot{\vartheta}'(\tau,\boldsymbol{\xi})\Big\}$$

Taking $\boldsymbol{\nabla}_\xi\cdot\boldsymbol{f}'(t,\boldsymbol{\xi})$ at the retarded time is not so straight forward because

$$[\boldsymbol{\nabla}_\xi\cdot\boldsymbol{f}'(t,\boldsymbol{\xi})]_\tau \neq \boldsymbol{\nabla}_\xi\cdot\boldsymbol{f}'(\tau,\boldsymbol{\xi})$$

Why is that? The difficulty is, that when we simply replace t by $\tau = \tau(t,r)$ then through r we introduce a new dependency on $\boldsymbol{\xi}$ and the divergence operator $\boldsymbol{\nabla}_\xi\cdot$ would not just act on the explicit dependence of $f'(\tau = t - r/a_\infty, \boldsymbol{\xi})$ on $\boldsymbol{\xi}$ (as the source term tells us), but also on its implicit dependence on $\boldsymbol{\xi}$ through r or τ respectively. But we may correct this appropriately when applying the chain-rule

$$\boldsymbol{\nabla}_\xi \cdot \boldsymbol{f}'(\tau, \boldsymbol{\xi}) = \left(\frac{\partial \boldsymbol{f}'}{\partial \tau} \cdot \boldsymbol{\nabla}_\xi \tau \right)_\xi + \underbrace{\left(\boldsymbol{\nabla}_\xi \cdot \boldsymbol{f}' \right)_\tau}_{[\boldsymbol{\nabla}_\xi \cdot \boldsymbol{f}'(t, \boldsymbol{\xi})]_\tau}$$

where the subscript ξ means "differentiation while keeping $\boldsymbol{\xi}$ fixed" and subscript τ means "differentiation while keeping τ fixed". We may now write the $\boldsymbol{\nabla}_\xi \cdot \boldsymbol{f}'$-part of Q_p in the integral (101) as

$$\int_{V_S} \frac{[\boldsymbol{\nabla}_\xi \cdot \boldsymbol{f}']_\tau}{r} dV(\boldsymbol{\xi}) = \int_{V_S} \overbrace{\left(\boldsymbol{\nabla}_\xi \cdot (r^{-1} \boldsymbol{f}') + r^{-2} \boldsymbol{f}' \cdot \boldsymbol{\nabla}_\xi r \right)}^{\frac{1}{r} \boldsymbol{\nabla}_\xi \cdot \boldsymbol{f}'(\tau, \boldsymbol{\xi})} dV(\boldsymbol{\xi}) - \int_{V_S} \frac{1}{r} \frac{\partial \boldsymbol{f}'}{\partial \tau} \cdot \boldsymbol{\nabla}_\xi \tau dV(\boldsymbol{\xi})$$

where the first integrand vanishes when we use Gauss' theorem for an infinitesimally enlarged integration volume on whose surface by definition $\boldsymbol{f}' \equiv \boldsymbol{0}$. Finally, with $\boldsymbol{\nabla}_\xi \tau = -a_\infty^{-1} \boldsymbol{\nabla}_\xi r$ we can write our solution (101) equivalently as

$$p'(\boldsymbol{x}, t) = \frac{1}{4\pi} \left\{ \int_{V_S} \frac{1}{r} \frac{\partial}{\partial t} \left[\left(1 - \frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty} \right) \dot{m}' + \frac{\sigma_\infty}{T_\infty} \dot{\vartheta}' + \frac{f_r'}{a_\infty} \right]_\tau dV(\boldsymbol{\xi}) + \int_{V_S} \frac{[f_r']_\tau}{r^2} dV(\boldsymbol{\xi}) \right\} \tag{103}$$

where $f_r' = -\boldsymbol{f}' \cdot \boldsymbol{\nabla}_\xi r$ denotes the component of the local force vector $\boldsymbol{f}'(\tau, \boldsymbol{\xi})$ in the direction of the observer. The respective solution in the frequency domain reads (note, $\dfrac{\partial}{\partial t}$ translates to a multiplication by $i\omega$)

$$\hat{p}(\boldsymbol{x}, \omega) = \frac{1}{4\pi} \left\{ \int_{V_S} \frac{i\omega \exp(-ikr_\xi)}{r} \left[\left(1 - \frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty} \right) \hat{m} + \frac{\sigma_\infty}{T_\infty} \hat{\vartheta} + \frac{\hat{f}_r}{a_\infty} \right] dV(\boldsymbol{\xi}) + \int_{V_S} \frac{\hat{f}_r \exp(-ikr)}{r^2} dV(\xi) \right\} \tag{104}$$

This formulation of the solution indicates that there are components of the solution slowly ($\sim 1/r$) and rapidly ($\sim 1/r^2$) decaying with distance. Note that the slowly decaying part of the forcing f_r' vanishes for incompressible fluids, because it is pre-multiplied by a_∞^{-1}. It occured in the above derivation solely due to the fact that the retarded time is (through a_∞) a function of the source point position. Moreover, through f_r' we observe an explicit dependence of the pressure field on the orientation $\boldsymbol{e}_r = -\boldsymbol{\nabla}_\xi r$ of the observer w.r.t. to the source, a so called "directivity".

There is yet another way in which this solution can be written, in fact the most commonly used form. It is obtained upon exchanging $\boldsymbol{\nabla}_\xi$ with $\boldsymbol{\nabla}_x$ through the relation $\boldsymbol{\nabla}_\xi r = -\boldsymbol{\nabla}_x r$. When we consider

$$\boldsymbol{\nabla}_x \cdot \int\limits_{V_S} \frac{\boldsymbol{f}'(\tau,\boldsymbol{\xi})}{r}\, dV(\xi) = \int\limits_{V_S} -\frac{1}{r a_\infty} \underbrace{\left[\frac{\partial \boldsymbol{f}'}{\partial \tau} \cdot \boldsymbol{\nabla}_x r\right]_\tau}_{= \frac{\partial [f_r']_\tau}{\partial \tau}} - \frac{1}{r^2} \underbrace{[\boldsymbol{f}' \cdot \boldsymbol{\nabla}_x r]_\tau}_{[f_r']_\tau}\, dV(\xi)$$

we retrieve the respective r.h.s.-terms in (103) which upon substitution yields equivalently

$$p'(\boldsymbol{x},t) = \frac{1}{4\pi}\left\{\frac{\partial}{\partial t}\int\limits_{V_S} \frac{1}{r}\left[\left(1-\frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty}\right)\dot{m}' + \frac{\sigma_\infty}{T_\infty}\dot{\vartheta}'\right]_\tau dV(\xi) - \boldsymbol{\nabla}_x \cdot \int\limits_{V_S} \frac{\boldsymbol{f}'(\tau,\boldsymbol{\xi})}{r}\, dV(\xi)\right\} \quad (105)$$

2.5 Mathematical tools

When considering more complicated source distributions, especially moving sources, it will become somewhat difficult to keep an overview about the mathematical manipulations. The use of generalized function theory will then prove to be an extremely helpful tool. Also the Green's function method for the solution of linear partial differential equations is an everyday's working tool for acousticians and will be introduced as well. The spatial expansion of sources into so called multipoles will help to classify and characterize different types of acoustic sources. In the aeroacoustic context we will be able to identify volume- and surface sources. Finally, we will introduce the reciprocity relation, which may conceptually be very helpful in reducing the complexity of acoustic sources.

2.5.1 Useful relations from generalized functions theory

We will not give a mathematical introduction into the theory of generalized functions. Here we give only a selection of the most often needed relations in the context of acoustic applications. A short introduction to generalized functions can be found e.g. in [1]. Generalized functions are functions which are defined by integral properties. They are not necessarily defined at each possible argument.

We begin our list of most commonly used generalized functions with the unit step function or so called "Heaviside function" $H(x)$

$$H(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \phantom{\text{for}} \quad x > 0 \end{cases} \quad (106)$$

Note, that the Heaviside function is not defined at $x = 0$. It appears that we will not need this function value. The most important generalized function is certainly the "Dirac function" or "Delta function" $\delta(x)$. We may define it through $H(x)$ by

$$\delta(x) = \frac{dH}{dx} \quad (107)$$

The Delta function is zero everywhere except at $x = 0$, where it is $\delta(0) = \infty$, such that $\int\limits_{-\infty}^{\infty} \delta(x)\, dx = 1$. When the product of δ and some "good" function $g(x)$ (a function in the ordinary sense) is integrated over an interval containing the position x_0, where the Delta function is non-zero, it acts as to "cut out" the value of g at the position x_0:

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = g(x_0) \quad \left\{ = \int_{<x_0}^{>x_0} g(x)\,\delta(x-x_0)\,dx \right\} \tag{108}$$

which may as well serve as the definition of the Delta function. In order to decide whether or not the Delta function is odd or even, we may test the outcome of integral (108) upon changing the sign of the argument of $\delta(x-x_0) \longrightarrow \delta(x_0-x)$. But a co-ordinate transform to $\overline{x} = -x$ then shows, that again the result is $g(x_0)$, which shows that $\delta(x)$ is an even function. As a consequence $\int_{<x_0}^{x_0} g(x)\,\delta(x-x_0)\,dx = \int_{x_0}^{>x_0} g(x)\,\delta(x-x_0)\,dx = g(x_0)/2$. Another consequence of (108) is

$$g(x)\,\delta(x-x_0) = g(x_0)\,\delta(x-x_0) \tag{109}$$

and if we specifically choose $g(x) = x$ and $x_0 = 0$ we have

$$x\,\delta(x) = 0 \cdot \delta(x) = 0 \tag{110}$$

One may use this relation to determine the derivatives of δ, i.e.

$$x^n\,\frac{d^n\delta}{dx^n} = (-1)^n n!\,\delta(x) \tag{111}$$

Note that (110) is also the reason why in the frame of generalized functions the inversion of a multiplication process is unique only up to a delta function:

$$x\,f(x) = 1 \quad \Longrightarrow \quad f(x) = x^{-1} + C\delta(x) \tag{112}$$

with C an arbitrary constant.

Another interesting property of the delta function is

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x) \tag{113}$$

which follows from the fact that δ is even, i.e. $\delta(\alpha x) = \delta(-\alpha x) = \delta(|\alpha| x)$ and the definitions (106, 107). Call $\overline{x} := |\alpha| x$. Then $\delta(\overline{x}) = dH(\overline{x})/d(\overline{x})$. But due to (106) $H(x) = H(\overline{x})$, while $d(\overline{x}) = |\alpha| dx$, i.e. $\delta(\overline{x}) = |\alpha|^{-1} dH/dx$ and thus (113).

A consequence of (113) is

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\,\delta(h(x))\,dx &= \sum_{x_i(h=0)} g(x_i)\left|\frac{dh}{dx}\right|_{x_i}^{-1} \quad \text{for} \quad \left(\frac{dh}{dx}\right)_{x_i} \neq 0 \qquad (114)\\
&= \sum_{x_i(h=0)} \int_{-\infty}^{\infty} g(x)\left|\frac{dh}{dx}\right|_{x_i}^{-1} \delta(x-x_i)\,dx \\
\Longrightarrow \delta(h(x)) &= \sum_{x_i(h=0)} \left|\frac{dh}{dx}\right|_{x_i}^{-1} \delta(x-x_i)
\end{aligned}$$

where the summation is meant over all zeros x_i of $h(x)$. This is immediately seen when considering the good function $h(x)$ near its zero x_i. Provided the first derivative of h at x_i is non-zero the Taylor expansion about x_i is $h(x) \simeq \left(\frac{dh}{dx}\right)_{x_i}(x - x_i) + \cdots$ and the identification of $\left(\frac{dh}{dx}\right)_{x_i}$ with α in (113) while appreciating (108) immediately yields the above relation.

Higher dimensional Delta functions are defined in an obvious way:

$$\delta(\boldsymbol{x}) = \delta(x)\delta(y)\delta(z) \tag{115}$$

in three dimensions. If $\delta(z)$ is replaced by 1 the two dimensional Delta function follows. We have the obvious generalization of (108)

$$\int\limits_{V_\infty} g(\boldsymbol{x})\,\delta(\boldsymbol{x} - \boldsymbol{x}_0)\,dV(\boldsymbol{x}) = g(\boldsymbol{x}_0) \tag{116}$$

where V_∞ denotes all space. A direct extension of (114) to more-dimensional (in fact also more than three-dimensional) integration yields the relation

$$\int\limits_{V_\infty} g(\boldsymbol{x})\,\delta(h(\boldsymbol{x}))\,dV = \sum_{S_i(h=0)} \int\limits_{S_i} g(\boldsymbol{x}_S)\frac{1}{|\boldsymbol{\nabla} h|_{\boldsymbol{x}_S}}\,dS \tag{117}$$

where h is some good scalar function, S_i is the i'th level surface $h = 0$ and $\boldsymbol{x}_S \in S_i$, i.e. $h(\boldsymbol{x}_S) \equiv 0$. The relation is obtained when considering $h(\boldsymbol{x})$ in the vicinity of $h(\boldsymbol{x}_S) = 0$. A Taylor expansion gives $h(\boldsymbol{x}) \simeq 0 + (\boldsymbol{\nabla} h)_{x_S}\boldsymbol{\cdot}(\boldsymbol{x} - \boldsymbol{x}_S) + \cdots$ Since the gradient of a scalar function is orthogonal to its level surfaces we find the unit normal vector to the level surface $h = 0$ at point x_S as $\boldsymbol{n} := (\boldsymbol{\nabla} h)_{x_S}/|\boldsymbol{\nabla} h|_{x_S}$. This may be used in the above Taylor expansion to yield $h(\boldsymbol{x}) \simeq |\boldsymbol{\nabla} h|_{x_S}(\boldsymbol{n\cdot x} - \boldsymbol{n\cdot x}_S) + \cdots = |\boldsymbol{\nabla} h|_{x_S}(x_n - x_{nS}) + \cdots$, where x_n obviously is the coordinate locally orthogonal to the level surface. Inserting h into the Delta function then gives $\delta(h) = \delta(|\boldsymbol{\nabla} h|_{x_i}(x_n - x_{ni})) = \delta(x_n - x_{ni})/|\boldsymbol{\nabla} h|_{x_i}$ according to (113). The volume element dV in (117) may be expressed as $dV = dx_n\,dS$, where dS is a surface element on the considered level surface. The integration over dx_n can now be carried out explicitly and it follows (117).

2.5.2 Green's function method

The Green's function method is to solve linear partial differential equations like our wave equation (61) in a very simple and formal way. In order to show the way in which this is done we first restrict ourselves to the free field problem. We want to solve the linear differential equation

$$L[p'] = Q_p(\boldsymbol{x}, t) \quad , \quad \text{with} \quad L = L(\boldsymbol{\nabla}_x, \frac{\partial}{\partial t}, \text{parameters}) \tag{118}$$

valid in the space $V = V_\infty$, which is all space. L is a differential expression in time and space. For our wave equation we have e.g. $L = a_0^{-2}\frac{\partial^2}{\partial t^2} - \Delta_x$.

Before solving (118) we solve the simplified problem

$$L[G] = \delta(\boldsymbol{x} - \boldsymbol{\xi})\delta(t - \tau) \tag{119}$$

where δ is the Delta function introduced above. Note that equation (119) describes the field p', or rather $G(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau)$, due to a unit point source at $\boldsymbol{x} = \boldsymbol{\xi}$, "firing" a needle pulse at $t = \tau$. G is called "Green's function" and it establishes the physical relation between two points $\boldsymbol{x}$ and $\boldsymbol{\xi}$ in space and in time t and τ. Upon multiplying (119) by $Q_p(\boldsymbol{\xi}, \tau)$ and integrating over all time and space V_∞ we have

$$\int\limits_{V_\infty} \int_{-\infty}^{t^+} L[G]\, Q_p(\boldsymbol{\xi}, \tau)\, d\tau\, dV(\xi) = \int\limits_{V_\infty} \int_{-\infty}^{t^+} \delta(\boldsymbol{x} - \boldsymbol{\xi})\delta(\tau - t) Q_p(\boldsymbol{\xi}, \tau)\, d\tau\, dV(\xi) \tag{120}$$

Why would we not integrate over all times $\int_{-\infty}^{\infty} d\tau$? For reasons of causality we want to make sure, that any quantity at the observer time t depends only on the past up to the present, in fact fully including the observer point in time, allowing $t^+ = \lim_{\varepsilon \to 0} t(1 + \varepsilon), \varepsilon > 0$. Looking again at (120) we recognize, that on the left there are some operations acting on the sources while according to the definition of the Delta function (108, 116) the right hand side represents the source $Q_p(\boldsymbol{x}, t)$ itself (even if we had integrated over all times). The source cannot anticipate its future and thus whatever operations we are performing on it, they must exclude terms, evaluated at $t > t^+$. We cannot automatically expect from a Green's function G to guarantee this important physical circumstance. Satisfaction of causality ($G(\tau > t^+) = 0$) is not intrinsic to them, even though this may very well be. Therefore it is in any case more "save" to extend the time integration at most over the past up to the present observer time, i.e. $\int_{-\infty}^{t^+} d\tau$. Next observe that due to (118) L is acting in $\boldsymbol{x}$ and t instead of $\boldsymbol{\xi}$ and τ, such that it can be taken outside of the integral on the left hand side. Moreover, in view of the definition of the Delta function (108, 116) the right hand side integrations yield nothing but $Q_p(\boldsymbol{x}, t)$, such that we obtain

$$L\underbrace{\Big[\int\limits_{V_\infty} \int_{-\infty}^{t^+} G(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau)\, Q_p(\boldsymbol{\xi}, \tau)\, d\tau\, dV(\xi)\Big]}_{= p'(\boldsymbol{x}, t) \quad \text{see (118)}} = Q_p(\boldsymbol{x}, t) \tag{121}$$

This means, that if the Green's function G to a problem is known (the solution due to a point-pulse source) the general solution p' due to some given source distribution $Q_p(\boldsymbol{x}, t)$ is obtained by explicit integration (convolution of the source with G). From (121) it is seen as well, that for a causal Green's function, which by definition satisfies $G(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \equiv 0$ for $\tau > t$, the upper limit of the time integration in (121) could have been equivalently extended to ∞, still garantueeing p' to be causal.

We recall that we have already solved the wave equation (61), where $L = a_\infty^{-2} \dfrac{\partial^2}{\partial t^2} - \Delta_x$, under free field conditions (no bodies present), see (101). We may extract the corresponding Green's function for the wave equation from it. This Green's function is then called "free field Green's function" G_0 (not tailored to specific boundary conditions). We re-write the free field pressure solution p' from (101) in the following way

$$p'(\boldsymbol{x}, t) = \frac{1}{4\pi} \int\limits_{V_S} \frac{Q_p(t - r/a_\infty, \boldsymbol{\xi})}{r} dV(\xi) = \int\limits_{V_\infty} \int_{-\infty}^{\infty} \frac{\delta(\tau - t + |\boldsymbol{x} - \boldsymbol{\xi}|/a_\infty)}{4\pi |\boldsymbol{x} - \boldsymbol{\xi}|} Q_p(\tau, \boldsymbol{\xi})\, d\tau\, dV(\xi)$$

where we have simply used (108) in order to replace $Q_p(t - r_\xi/a_\infty)$ by $\int_{-\infty}^{\infty} Q_p(\tau)\delta(\tau - t + r_\xi/a_\infty)d\tau$ which in turn equals $\int_{-\infty}^{t^+} Q_p(\tau)\delta(\tau - t + r_\xi/a_\infty)d\tau$ because our Delta function satisfies causality $\delta(\tau > t) = 0$. Also we have taken the liberty to extend the spatial integration to all space V_∞ because Q_p is zero outside $V_S \in V_\infty$ anyways. A direct comparison with (121) now yields the free field Green's function for the wave equation (61) to be:

$$G_0(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) = G_0(\boldsymbol{x} - \boldsymbol{\xi}, \tau - t) = \frac{\delta(\tau - t + |\boldsymbol{x} - \boldsymbol{\xi}|/a_\infty)}{4\pi|\boldsymbol{x} - \boldsymbol{\xi}|} \tag{122}$$

For obvious reasons our Green's function due to (122) satisfies $G_0 \equiv 0$ for $\tau > t$, such that the pressure field p' computed by convolving G_0 with a given source Q_p according to

$$p'(\boldsymbol{x}, t) = \int\limits_{V_\infty} \int_{-\infty}^{\infty} G_0\, Q_p(\tau, \boldsymbol{\xi})\, d\tau\, dV(\boldsymbol{\xi}) = \frac{1}{4\pi} \int\limits_{V_S} \frac{Q_p(t - r/a_\infty, \boldsymbol{\xi})}{r} dV(\boldsymbol{\xi}) \tag{123}$$

will satisfy the causality condition.

In the same way as we deduced the free field Green's function G_0 for the wave equation we may deduce the corresponding Green's function $\hat{G}_0$ for the Helmholtz equation. It satisfies by definition the equation $\Delta\hat{G}_0 + k^2\hat{G}_0 = \delta(\boldsymbol{x} - \boldsymbol{\xi})$. We extract $\hat{G}_0$ from (102)

$$\hat{G}_0(\boldsymbol{x} - \boldsymbol{\xi}, \omega) = \frac{\exp(-ik|\boldsymbol{x} - \boldsymbol{\xi}|)}{4\pi|\boldsymbol{x} - \boldsymbol{\xi}|} \tag{124}$$

(For the free field Green's function in other dimensions and for other equations see table *B.1*). Note that $\hat{G}_0$ satisfies the requirement of outgoing waves radiation condition (82), being the analogue of the causality condition on G_0 in real space.

2.6 Acoustics in stagnant homogeneous media – part 2 –

After having introduced some mathematical tools, we proceed with concepts which have become common in acoustics. Their brief discussion is important as below we want to use these concepts also in the context of aeroacoustics.

2.6.1 Pressure field in the presence of obstacles

In the preceeding sections we have not explicitly taken into account the presence of objects (bodies) in the pressure field p'. Before doing so a first notion of the influence of a boundary may be obtained when simply appreciating the following. First, a source region V_S is defined by the fact, that the pressure p' at a source point $\boldsymbol{x} \in V_S$ does not satisfy the homogeneous wave equation (61) with $Q_p = 0$, describing the source-free wave propagation. In exactly the same way we may say that at a boundary we require some boundary condition for the pressure p' instead of the satisfaction of the homogeneous wave equation, which for that matter will be violated just

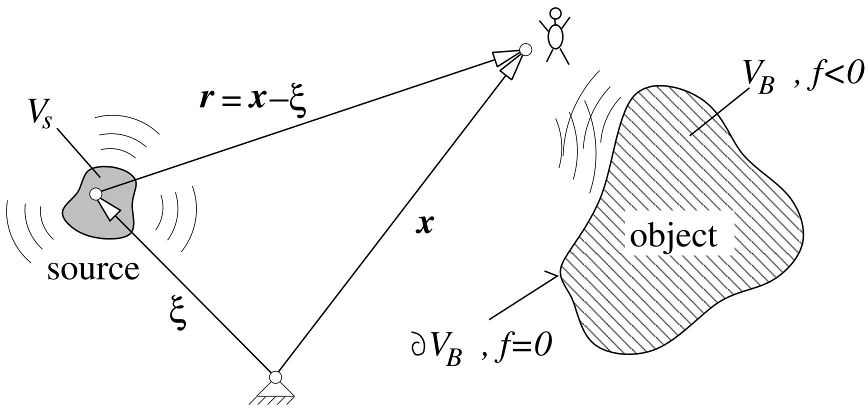

Figure 15: Sound field with obstacles.

as on a source point. Therefore formally a boundary may be regarded as nothing but a source concentrated on ∂V_B, i.e. some singular source distribution. We will make this more explicit in the following section. Thereafter we introduce the concept of tailored Green's functions and finally define acoustic surface properties in order to include them into the solution.

We include the presence of obstacles by applying the tools of generalized function theory. First we identify a body occupying the volume V_B by some function $f(\boldsymbol{x}, t)$, whose zero-contour line is identical with the body boundary for all times t (see figure *15*):

$$\begin{array}{lll} f(\boldsymbol{x},t) < 0 & & \boldsymbol{x} \in V_B \\ f(\boldsymbol{x},t) = 0 & \text{for} & \boldsymbol{x} \in \partial V_B \\ f(\boldsymbol{x},t) > 0 & & \boldsymbol{x} \in V_\infty \backslash \{V_B \cup \partial V_B\} \end{array} \tag{125}$$

where V_∞ is again all of space. Now we account for the presence of the body simply by introducing the new pressure variable

$$\underline{p}' := H(f) p' \tag{126}$$

and from the definition of the Heaviside function H it is clear, that in the field $\boldsymbol{x} \in V_\infty \backslash V_B$ our new variable corresponds to the pressure $\underline{p}' := p'$, while inside the body $\underline{p}' \equiv 0$, requiring it to jump in value at the boundary. The very essential step here is, that while p' is defined only outside the body and on its boundaries, $\underline{p}'$ is defined in all space V_∞. This effectively reduces the boundary value problem to the free field problem, whose Green's function G_0 for the wave equation we know already (122). Let us now apply the wave operator $L = a_\infty^{-2} \dfrac{\partial^2}{\partial t^2} - \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ on the left and right of (126). Certainly we have to apply the product rule on the r.h.s. and the chain rule on the Heaviside function $\boldsymbol{\nabla} H = \dfrac{dH}{df} \boldsymbol{\nabla} f = \delta(f) \boldsymbol{\nabla} f$ and $\dfrac{\partial H}{\partial t} = \dfrac{dH}{df} \dfrac{\partial f}{\partial t} = \delta(f) \dfrac{\partial f}{\partial t}$ successively to obtain

$$L[\underline{p}'] = \underbrace{H(f) L[p']}_{H(f) Q_p} + \frac{1}{a_\infty^2} \left\{ \frac{\partial}{\partial t} \left[p' \delta(f) \frac{\partial f}{\partial t} \right] + \delta(f) \frac{\partial p'}{\partial t} \frac{\partial f}{\partial t} \right\} - \boldsymbol{\nabla} \cdot \left[p' \delta(f) \boldsymbol{\nabla} f \right] - \delta(f) \boldsymbol{\nabla} p' \cdot \boldsymbol{\nabla} f \tag{127}$$

The above equation equals our wave equation (61) except the four extra terms, which are zero except on the boundary at $f = 0$. Next we need to express the time and space derivative of f. Due to its derivation our wave equation is governing the dynamics of small perturbations p' about a constant mean value p^0. This restricts the motion of the object $f = 0$ to stay in the frame of small perturbations as well, since for large velocities it would necessarily generate large pressure fluctuations. Therefore we may only allow for small perturbations $\varepsilon f'(\boldsymbol{x}, t)$ in the boundary shape $f^0(\boldsymbol{x})$, i.e. $f = f^0 + \varepsilon f'$. The time derivative of f in (127) is consequently $\dfrac{\partial f}{\partial t} = \varepsilon \dfrac{\partial f'}{\partial t}$ and vanishes as $\varepsilon \to 0$ according to the linear approximation. The gradient is $\boldsymbol{\nabla} f = \boldsymbol{\nabla} f^0 + \varepsilon \boldsymbol{\nabla} f'$, i.e. $\lim\limits_{\varepsilon \to 0} \boldsymbol{\nabla} f = \boldsymbol{\nabla} f^0$, which is nothing but a vector pointing normal to the contour lines and increasing values of the scalar function f^0 at point $\boldsymbol{x}$. Therefore $\boldsymbol{n} := (\boldsymbol{\nabla} f / |\boldsymbol{\nabla} f|)_{f=0}$ represents the unit outward normal to the boundary.

Let us now solve (127) for $\underline{p}'$ by making use of the general Green's function solution procedure for free field problems: i) multiply by G and ii) integrate over all space V_∞ and time:

$$\underline{p}'(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \int\limits_{V_\infty} \Big\{ H(f^0)\, Q_p \underbrace{- \boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \Big[p' \delta(f^0) \boldsymbol{\nabla}_\xi f^0 \Big] - \delta(f^0) \boldsymbol{\nabla}_\xi\, p' \boldsymbol{\cdot} \boldsymbol{\nabla}_\xi f^0 \Big\} G}_{=:\, I} \; dV(\boldsymbol{\xi})\, d\tau$$

Of this, the last two terms can be written more explicitly:

$$I = -\boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \Big[p' \delta(f^0) \boldsymbol{\nabla}_\xi f^0\, G \Big] + \underbrace{p' \delta(f^0) \boldsymbol{\nabla}_\xi f^0 \boldsymbol{\cdot} \boldsymbol{\nabla}_\xi\, G - \delta(f^0)(\boldsymbol{\nabla}_\xi\, p' \boldsymbol{\cdot} \boldsymbol{\nabla}_\xi f^0) G}_{\delta(f^0) |\boldsymbol{\nabla}_\xi f^0|\, (p' \boldsymbol{n}^0 \boldsymbol{\cdot} \boldsymbol{\nabla}_\xi G - G \boldsymbol{n}^0 \boldsymbol{\cdot} \boldsymbol{\nabla}_\xi\, p')}$$

When I is integrated over the space V_∞ the first (divergence) part vanishes upon transforming it to a surface integral over the bounding surface of V_∞, where $\delta(f^0) \equiv 0$ (we assume the body to be of finite extent). The second part is -as shown in the underbrace- proportional to $\delta(f^0)|\boldsymbol{\nabla}_\xi f^0|$ and use of the relation (117) transfers the space integration to the surface(s) S, defined by $f^0 = 0$. These manipulations finally yield the pressure solution in the presence of surfaces:

$$\underline{p}'(\boldsymbol{x}, t) = H(f^0)\, p'(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} \Big\{ \int\limits_{V_S} H(f^0)\, Q_p\, G\; dV(\boldsymbol{\xi}) + \int\limits_{\partial V_B : f^0 = 0} \Big(p' \frac{\partial G}{\partial n} - G \frac{\partial p'}{\partial n} \Big) dS(\boldsymbol{\xi}) \Big\}\, d\tau \tag{128}$$

where $\dfrac{\partial}{\partial n} := \boldsymbol{n}^0 \boldsymbol{\cdot} \boldsymbol{\nabla}$ denotes the normal derivative on ∂V_B (or $f^0 = 0$), directed outside of V_B. Also, since the source Q_p is different from zero only in the finite volume V_S we replaced V_∞. Note that we have to assume a causal Green's function which e.g. would be ensured by G_0 from (122). Upon Fourier transforming (128) to the frequency domain the solution reads

$$\underline{\hat{p}}(\boldsymbol{x}, \omega) = H(f^0)\, \hat{p}(\boldsymbol{x}, \omega) = \int\limits_{V_S} H(f^0)\, \hat{Q}_p\, \hat{G}\; dV(\boldsymbol{\xi}) + \int\limits_{\partial V_B : f^0 = 0} \Big(\hat{p} \frac{\partial \hat{G}}{\partial n} - \hat{G} \frac{\partial \hat{p}}{\partial n} \Big) dS(\boldsymbol{\xi}) \tag{129}$$

where $\hat{G}$ could be taken from (124). Note that in this case we do not have to worry about the physical requirement of outgoing waves, because $\hat{p}$ was obtained from the causal solution p' (not allowing for waves coming from infinity). Alternatively we may also say that $\hat{G}_0$ from (124) satisfies the outgoing wave (or radiation) condition.

A still more explicit form of the solution is obtained by inserting the explicit expression for the free field Green's function into (128) as listed in table (*B.1*) for the 1D, 2D and 3D case.

In 3D this is for instance $G_0 = \delta(g)/(4\pi r)$ with $g := \tau - t + r/a_\infty$. With the help of the known expression for G_0 we may actually evaluate the time integral in (128). In order to do so, the integral $\int_{-\infty}^{\infty} p' \frac{\partial G_0}{\partial n}\, d\tau$ is slightly re-arranged. We re-formulate $p'\frac{\partial G_0}{\partial n} = p'\frac{\partial G_0}{\partial r}\frac{\partial r}{\partial n} = \frac{p'}{4\pi}\Big(\frac{d\delta}{dg}\frac{\partial g}{\partial r}\frac{1}{r} - \frac{\delta}{r^2}\Big)\frac{\partial r}{\partial n} = \frac{p'}{4\pi}\Big(\frac{d\delta}{d\tau}\frac{1}{a_\infty r} - \frac{\delta}{r^2}\Big)\frac{\partial r}{\partial n}$. Note that by definition $\frac{\partial r}{\partial n} = \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\nabla}_\xi r = -\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\nabla}_x r = -\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{e}_r$ with $\boldsymbol{e}_r = \boldsymbol{r}/r$. Next we use the product rule to write the first term like $p'\frac{d\delta}{d\tau}\frac{1}{a_\infty r} = \frac{1}{a_\infty r}\Big(\frac{\partial p'\delta}{\partial \tau} - \delta\frac{\partial p'}{\partial \tau}\Big)$. The integration over τ of the first of these two terms gives a zero because we require that a physical signal cannot have existed for all times, i.e. $p'(\tau = -\infty) = 0$. Now the time integrations of all other terms can be carried out, leading to their evaluation at the retarded time

$$\begin{aligned}\underline{p}'(\boldsymbol{x},t) &= H(f^0)\,p'(\boldsymbol{x},t) = \\ &= \int\limits_{V_S} H(f^0)\,\frac{Q_p(\tau,\boldsymbol{\xi})}{4\pi r}\,dV(\xi) + \int\limits_{\partial V_B:f^0=0} \frac{1}{4\pi r}\Big[\Big(\frac{1}{a_\infty}\frac{\partial p'}{\partial \tau} + \frac{p'}{r}\Big)\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{e}_r - \frac{\partial p'}{\partial n}\Big]_\tau dS(\xi)\end{aligned} \quad (130)$$

where again $\tau = t - r/a_\infty$ denotes the retarded time and $\boldsymbol{e}_r$ the direction along the distance vector $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{\xi}$. In the spectral space this reads:

$$\begin{aligned}&\underline{\hat{p}}(\boldsymbol{x},\omega) = H(f^0)\,\hat{p}(\boldsymbol{x},\omega) = \\ &\quad= \int\limits_{V_S} H(f^0)\,\frac{\hat{Q}_p \exp(-ikr)}{4\pi r}\,dV(\boldsymbol{\xi}) + \int\limits_{\partial V_B:f^0=0} \frac{\exp(-ikr)}{4\pi r}\Big[(ikr+1)\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{e}_r\frac{\hat{p}}{r} - \frac{\partial \hat{p}}{\partial n}\Big]dS(\boldsymbol{\xi})\end{aligned} \quad (131)$$

Unfortunately with (128) we have not really arrived at a true solution to the inhomogeneous wave equation (61) in the presence of boundaries ∂V_B yet, because the right hand side of our equation still depends on the solution p'. Note, that we may choose the observer position $\boldsymbol{x}$ on the boundary and thus obtain an integral equation for p', but there exists a second unknown in that equation, namely $\frac{\partial p'}{\partial n}$ on the boundary, calling for an extra equation. This boundary condition needs to be specified on ∂V_B:

$$L_B[p'] := A[p'] + B[\frac{\partial p'}{\partial n}] = C(\boldsymbol{x}_s,t) \quad , \quad \boldsymbol{x}_s \in \partial V_B \quad (132)$$

or in the frequency domain:

$$L_B[\hat{p}] = \hat{A}(\boldsymbol{x}_s,\omega)\hat{p} + \hat{B}(\boldsymbol{x}_s,\omega)\frac{\partial \hat{p}}{\partial n} = \hat{C}(\boldsymbol{x}_s,\omega) \quad , \quad \boldsymbol{x}_s \in \partial V_B \quad (133)$$

The boundary condition (132) and equation (128) evaluated for $\boldsymbol{x} = \boldsymbol{x}_s$ represent a closed boundary integral equation system for the pressure and its normal derivative on the surface. Such systems are usually solved numerically with the so called boundary element method (BEM), which will be discussed in the second part of the lecture. Once the pressure and its normal derivative are known on the boundary, (128) allows to explicitly determine the sound pressure at any chosen location $\boldsymbol{x}$. In the spectral space the respective system to solve is (129) and (133) on $\boldsymbol{x} = \boldsymbol{x}_s$.

The task remains to fill the formal expressions in (132) or (133) with a physical meaning, which is done in the following section on surface properties (2.6.2). But first we look at the case when a Green's function can be found which by itself already satisfies the boundary conditions of the problem.

2.6.1.1 Tailored Green's functions. Suppose we managed to find a so called *exact Green's function* $\hat{G}_{ex}$ satisfying the homogeneous version of (133), i.e. $L_B[\hat{G}_{ex}] = 0$. Then $\frac{\partial \hat{G}_{ex}}{\partial n} = -\frac{\hat{A}}{\hat{B}}\hat{G}_{ex}$ or alternatively $\hat{G}_{ex} = -\frac{\hat{B}}{\hat{A}}\frac{\partial \hat{G}_{ex}}{\partial n}$. Using these relations the integrand of the surface integral in (129) may be expressed like $\hat{p}\frac{\partial \hat{G}_{ex}}{\partial n} - \hat{G}_{ex}\frac{\partial \hat{p}}{\partial n} = -\frac{\hat{C}}{\hat{B}}\hat{G}_{ex} = \frac{\hat{C}}{\hat{A}}\frac{\partial \hat{G}_{ex}}{\partial n}$. Therefore for an exact (causal) Green's function the equation (129) represents explicitly the pressure field. No boundary integral equation must be solved then.

Suppose we managed to find a Green's function G_N, satisfying $\frac{\partial G_N}{\partial n} = 0$ on ∂V_B. This Green's function will then allow us to solve the von Neumann problem ($\frac{\partial p'}{\partial n}$ given on ∂V_B) by explicit integration of the r.h.s of (128), without having to solve an integral equation. For the "acoustically hard" surface, defined by $\frac{\partial p'}{\partial n} = 0$, the surface integral vanishes (assuming the use of G_N).

Conversely, suppose we found a Green's function G_D, satisfying $G_D = 0$ on ∂V_B. Then we have the explicit solution to the Dirichlet problem, i.e. the one where p' is given on the boundary ∂V_B. Again a limiting case is the "acoustically soft" surface or "pressure release surface" for which $p' = 0$. Note that this case would correspond to a locally reacting wall with an impedance $z = 0$.

Such special Green's functions, satisfying the boundary condition of the problem are called "Tailored Green's functions". They are of obvious computational advantage. If tailored Green's functions are not available as closed form analytical expression, they may be computed numerically using e.g. the Boundary Element Method.

2.6.2 Acoustic surface properties

The acoustic property of a surface may be understood as transfer function between an external load to the surface and its resulting dynamic behavior. Typically the (passive) surface property is specified in the frequency domain as (frequency dependent) *wall impedance* $\hat{z}(\omega)$ (German: "Wandimpedanz") or as *wall admittance* $\hat{a}(\omega) = 1/\hat{z}$ (German: "Wandadmittanz").

$Re(\hat{z})$ is called *acoustic resistance* (German: "akustischer Widerstand")
$Im(\hat{z})$ is called *acoustic reactance*(German: "akustische Reaktanz")

The impedance $\hat{z}$ or admittance $\hat{a}$ relate pressure and normal velocity component $\hat{v}_n^p(\boldsymbol{\xi}_S, \omega)$ for a given frequency ω.

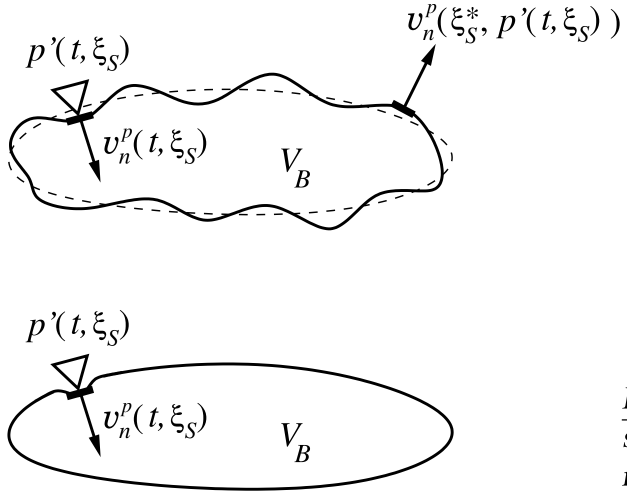

Figure 16: Surface response to an unsteady pressure load p' at surface position ξ_S. Top: globally reacting wall, bottom: locally reacting wall.

$$\hat{p}(\boldsymbol{\xi}_S,\omega) \quad =: \quad \hat{z}(\boldsymbol{\xi}_S,\omega)\,\hat{v}_n^p(\boldsymbol{\xi}_S,\omega) \tag{134}$$
$$\hat{v}_n^p(\boldsymbol{\xi}_S,\omega) \quad =: \quad \hat{a}(\boldsymbol{\xi}_S,\omega)\,\hat{p}(\boldsymbol{\xi}_S,\omega) \tag{135}$$

where $\boldsymbol{\xi}_S$ is a point on the boundary $S(\boldsymbol{\xi}_S) \equiv 0$ and $\hat{p}$ and $\hat{v}_n^p$ are the Fourier transforms of the pressure disturbance p' at the wall and the reaction to it, namely the wall normal velocity component v_n'. As a consequence of (134, 135) and physical realizability $\hat{z}(i\omega) = \hat{z}^*(-i\omega)$ (asterisk denoting complex conjugate) and $\hat{a}(i\omega) = \hat{a}^*(-i\omega)$, assuming real frequencies ω. For general cases the impedance is very difficult to determine, because it would be a function of the angle of incidence, at which a plane sound wave hits the surface. Also a deformable wall will usually respond with different types of global vibrational waves to incoming pressure disturbances, see top of fig. *16*. Then a true fluid/structure coupling takes place and we call the surface a *globally reacting wall* (German: "global reagierende Wand"). For the sake of simplicity we exclude such surface behavior here. We consider only the simplest of all cases, the so called *locally reacting wall* (German: "lokal reagierende Wand"). The assumption is, that the kinematic response of a surface element $dS(\boldsymbol{\xi}_S)$ at a surface point $\boldsymbol{\xi}_S$ is only a function of the locally applied force $\hat{p}(\boldsymbol{\xi}_S)dS(\boldsymbol{\xi}_S)$. The impedance is then independent of the angle of incidence and may be considered a function of surface position $\boldsymbol{\xi}_S$ and frequency ω only. Often, porous surfaces satisfy the assumption of a locally reacting wall.

The impedance and admittance may be expressed in the real space as well, when recognizing that according to definition (134) and the inverse Fourier transform (24) we have

$$p'(\boldsymbol{\xi}_S,t) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{z}(\omega)\,\hat{v}_n^p(\boldsymbol{\xi}_S,\omega)\exp(+i\omega t)\,d\omega = \int_{-\infty}^{\infty} z(t-\tau)v_n^p(\boldsymbol{\xi}_S,\tau)\,d\tau \tag{136}$$
$$v_n^p(\boldsymbol{\xi}_S,t) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{a}(\omega)\,\hat{p}(\boldsymbol{\xi}_S,\omega)\exp(+i\omega t)\,d\omega = \int_{-\infty}^{\infty} a(t-\tau)p'(\boldsymbol{\xi}_S,\tau)\,d\tau \tag{137}$$

where the respective last equality in (136) and (137) is due to the so called "convolution theorem". Note for causality reasons the pressure p' may only depend on the history but not the future of v_n^p.

Therefore $z(\tau > t) = 0$ is a requirement on z and $a(\tau > t) = 0$ respectively.

We may now finally attach some physcial information to the terms of the formal boundary relations (132) and (133) for the solution of the wave equation. These relate the pressure to the wall normal derivative of the pressure. The surface properties on the other hand relate the pressure and the wall normal velocity component. Therefore we still need to link the wall normal pressure derivative to the normal velocity component. This is possible thanks to the linearized momentum equation (58), multiplied by the wall normal vector $\boldsymbol{n}^0$ leaving

$$\frac{\partial p'}{\partial n} = -\rho^0 \frac{\partial v_n'}{\partial t} \quad \text{or} \quad \frac{\partial \hat{p}}{\partial n} = -i\omega\rho^0 \hat{v}_n \tag{138}$$

in real or frequency domain respectively. We neglected the presence of external forces $\boldsymbol{f}'$ on the boundary. Next, let us assume a quite general case, in which the surface is actively vibrating and at the same time subject to an external acoustic field. Then v_n' is composed of the given vibration of the surface v_n^{vib} and the responding motion v_n^p due to an external pressure load p':

$$v_n' = v_n^{vib} + v_n^p \quad \text{or} \quad \hat{v}_n = \hat{v}_n^{vib} + \hat{v}_n^p$$

We first express the boundary conditions in the spectral space (133). Insertion of $\hat{v}_n$ into (138) leaves $i\omega\rho^0\hat{v}_n^{vib} + i\omega\rho^0\hat{v}_n^p = -\frac{\partial \hat{p}}{\partial n}$. Herein we substitute $\hat{v}_n^p = \hat{a}\hat{p}$ according to (135) to finally obtain

$$\underbrace{i\omega\rho^0\hat{a}}_{\hat{A}}\,\hat{p} + \underbrace{1}_{\hat{B}}\,\frac{\partial \hat{p}}{\partial n} = \underbrace{-i\omega\rho^0\hat{v}_n^{vib}}_{\hat{C}} = L_B[\hat{p}]$$

Next we express the boundary conditions in the real space (132). From (138) we have $\rho^0\frac{\partial v_n^{vib}}{\partial t} + \rho^0\frac{\partial v_n^p}{\partial t} = -\frac{\partial p'}{\partial n}$. Now, the time change of the pressure induced velocity v_n^p may be determined from (137):

$$\frac{\partial v_n^p}{\partial t} = \int_{-\infty}^{\infty} \frac{\partial a}{\partial t}(t-\tau^*)p'(\boldsymbol{\xi}_S, \tau^*)\, d\tau^*$$

This finally gives

$$\underbrace{\rho^0 \int_{-\infty}^{\infty} \frac{\partial a}{\partial t}(t-\tau^*)p'(\boldsymbol{\xi}_S, \tau^*)\, d\tau^*}_{A[p']} + \underbrace{\frac{\partial p'}{\partial n}}_{B[\frac{\partial p'}{\partial n}]} = \underbrace{-\rho^0\frac{\partial v_n^{vib}}{\partial t}}_{C} = L_B[p'] \tag{139}$$

Let us finally mention two special, in applications often found cases of surface properties, namely

(a) *acoustically hard surface* (German: ”ideal schallharte Oberfläche”), for which $\hat{a} \equiv 0$ (or $\hat{z} \to \infty$). For a passive surface $C = 0$. Typically the surface of an object made of concrete may be considered an acoustically hard surface, $\frac{\partial p'}{\partial n} = 0$ on ∂V_B,
(b) *pressure release surface* (or acoustically soft surface) (German: ”ideal schallweiche Oberfläche”), for which $\hat{z} \equiv 0$ (or $\hat{a} \to \infty$). For a passive surface $C = 0$, $p' = 0$ on ∂V_B. An example for a pressure release surface is the free interface between water and air, when sound propagating in the water hits this surface. Then the particles at the surface are free to move out according to the incident acoustic particle velocity field.

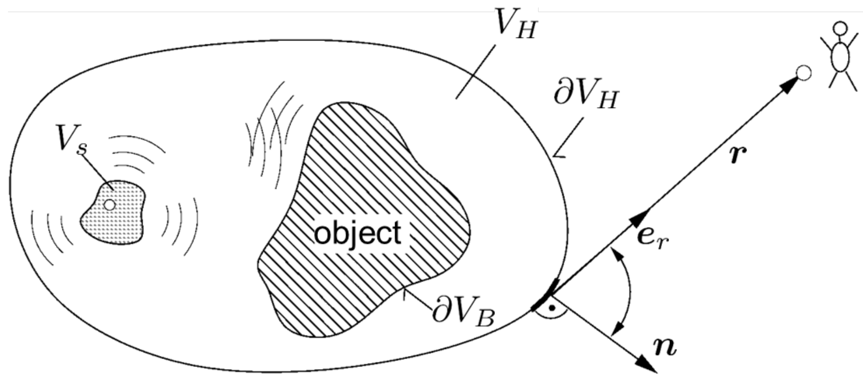

Figure 17: Kirchhoff integration surface ∂V_H.

2.6.3 Kirchhoff integral

If the pressure and its derivatives are known on an arbitrarily shaped closed surface ∂V_H (see figure 17), enclosing all source domains V_S and all objects V_B, then the sound pressure field on all locations outside ∂V_H may be determined with the Kirchhoff integral.

The Kirchhoff integral may be considered a special case of the relations (130) or (131), in which the integration surface does not only contain the objects V_B, but the whole arrangement of sources and objects. Analogously to (125) we define the domain V_H inside of ∂V_H with the help of the scalar function $f(\boldsymbol{x})$, whose zero-level surface is identical with ∂V_H:

$$\begin{array}{lll} f(\boldsymbol{x}) < 0 & & \boldsymbol{x} \in V_H \\ f(\boldsymbol{x}) = 0 & \text{for} & \boldsymbol{x} \in \partial V_H \\ f(\boldsymbol{x}) > 0 & & \text{else} \end{array} \tag{140}$$

Evaluating (130) for V_H instead of V_B eliminates the volume integral over the sources V_s because these source domains are located inside V_H.

$$\underline{p}' = \frac{1}{4\pi} \int\limits_{\partial V_H} \frac{1}{r} \Big(\frac{1}{a_\infty} \frac{\partial p'}{\partial \tau} + \frac{p'}{r} \Big) \boldsymbol{n} \cdot \boldsymbol{e}_r - \frac{1}{r} \frac{\partial p'}{\partial n} \, dS(\xi), \tag{141}$$

The Kirchhoff integral is written for a closed integration surface at rest and a medium at rest. The observer at $\boldsymbol{x}$ is at rest as well. The normal vector $\boldsymbol{n}$ on the surface element $dS(\xi)$ is by definition pointing towards the exterior of the surface; the unit vector from source element to observer is $\boldsymbol{e}_r = (\boldsymbol{x} - \boldsymbol{\xi})/r$. Note, that the integrals need to be evaluated at the retarded time $\tau = t - r/a_\infty$.

2.6.4 Expansion of sources into multipoles

We have seen, that there are ways to solve for the acoustic pressure, once the source term Q_p is known. In order to learn more about a sound source it is worthwile to look into its structure and try to characterize "types of sources". This introduces the so called "multipole expansion" of a source, so-to-speak, breaking it up into simple components. Moreover it will help to reduce sources to their dominant part and thus simplify the description considerably. The multipole expansion rests upon a spatial Taylor expansion of the Green's function $G(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau)$.

Suppose we have an exact Green's function (121) to a problem (satisfying differential equation and boundary conditions) as sketched in figure *15*. Given some source Q_p we obtain the solution through integration of $Q_p(\boldsymbol{\xi}, \tau)$ weighted by $G(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau)$ over the source region and time. The Green's function establishes the physical (in our case acoustic) relation between the two points $\boldsymbol{x}$, $\boldsymbol{\xi}$ and the moments t, τ. As before we consider $\boldsymbol{x}$ and t to be observer position and reception time respectively. The Green's function "acoustically maps" the source information onto the observation point. We Taylor-expand $G(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau)$ about some position $\boldsymbol{\xi}_0$, centred inside $V_S(\boldsymbol{\xi})$, i.e. the source volume. The terms of increasing order map different "acoustic features" or characteristics of the source Q_p to the observer point.

$$
\begin{aligned}
G(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) &= G(\boldsymbol{x}, t; \boldsymbol{\xi}_0, \tau) + (\boldsymbol{\nabla}_\xi G)_0 \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_0) + \frac{1}{2!}(\boldsymbol{\nabla}_\xi \boldsymbol{\nabla}_\xi G)_0 : (\boldsymbol{\xi} - \boldsymbol{\xi}_0)(\boldsymbol{\xi} - \boldsymbol{\xi}_0) + \frac{1}{3!} \cdots \quad (142) \\
&= G(\boldsymbol{x}, t; \boldsymbol{\xi}_0, \tau) + \left.\frac{\partial G}{\partial \xi_j}\right|_0 (\xi_j - \xi_j^0) + \frac{1}{2!} \left.\frac{\partial^2 G}{\partial \xi_j \partial \xi_k}\right|_0 (\xi_j - \xi_j^0)(\xi_k - \xi_k^0) + \frac{1}{3!} \cdots
\end{aligned}
$$

Due to (121) the solution $p'(\boldsymbol{x}, t)$ then appears correspondingly as

$$
\begin{aligned}
p'(\boldsymbol{x}, t) &= \int_{-\infty}^{t^+} G(\boldsymbol{x}, t; \boldsymbol{\xi}_0, \tau) \int_{V_S} Q_p(\boldsymbol{\xi}, \tau) dV(\xi)\, d\tau + \\
&+ \int_{-\infty}^{t^+} (\boldsymbol{\nabla}_\xi G)_0 \cdot \int_{V_S} (\boldsymbol{\xi} - \boldsymbol{\xi}_0) Q_p(\boldsymbol{\xi}, \tau) dV(\xi)\, d\tau + \\
&+ \int_{-\infty}^{t^+} (\boldsymbol{\nabla}_\xi \boldsymbol{\nabla}_\xi G)_0 : \int_{V_S} \frac{1}{2!} (\boldsymbol{\xi} - \boldsymbol{\xi}_0)(\boldsymbol{\xi} - \boldsymbol{\xi}_0) Q_p(\boldsymbol{\xi}, \tau) dV(\xi)\, d\tau + \\
&+ \ \ldots \qquad (143)
\end{aligned}
$$

which shows that the volume integrals represent a sequence of differently weighted averages of the source function over the source domain. In this way the source region is lumped together and reduced to the point $\boldsymbol{\xi}_0$. The first term in the above sequence is called "monopole term", the second "dipole term" the third "quadrupole term". In general, if m denotes the order of the term in the Taylor expansion (first term $m = 0$, second term $m = 1$ etc.), then the corresponding pole is called "pole of the order 2^m". For instance the dipole term is also called pole of order 2. We emphasize, that the appearance of say, the quadrupole term of the same source Q_p may be completely different, depending on the Green's function used for the expansion. The quadrupole term of a source Q_p in the free field (Green's function G_0) has completely different characteristics than the quadrupole term of the same source near, say, a sharp edge of a given geometry (tailored Green's function taking account of the boundary conditions). This indeed has far reaching consequences in the description of aeroacoustic sources, where (see below) the aerodynamic source function corresponds in nature to a quadrupole-type source. The noise from free turbulence has most different character compared to the noise produced by the same turbulence near e.g. a trailing edge.

Of particular interest is the multipole expansion w.r.t. the free field Green's function G_0 due to (122), because we may do our expansion explicitly. It is now more convenient to write the expansion in index notation $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\boldsymbol{x} = (x_1, x_2, x_3)$. We introduce the superscripts i, j and k to denote the number of differentiations w.r.t. the co-ordinate directions 1, 2, and 3 respectively. The $i+j+k$'th expansion coefficient then contains the following $\boldsymbol{\nabla}_\xi$-differentiations of $G_0(|(x_i - \xi_i)|, \tau - t)$ at $\boldsymbol{\xi} = \boldsymbol{\xi}_0$ (no summation over i, j, k):

$$\frac{\partial^{i+j+k} G_0(|\boldsymbol{x} - \boldsymbol{\xi}|)}{\partial \xi_1^i \partial \xi_2^j \partial \xi_3^k}\Big|_{\xi_0} = (-1)^{i+j+k} \frac{\partial^{i+j+k}}{\partial x_1^i \partial x_2^j \partial x_3^k}[G_0(|\boldsymbol{x} - \boldsymbol{\xi}_0|)]$$

As indicated, the sign-factor provides a minus whenever $i + j + k$ is odd, because each single exchange from ξ to x is accompanied with a change in sign due to the above mentioned dependence of G_0 on $|(x_i - \xi_i)|$. We may now insert G_0 into the Taylor expansion of the solution

$$p'(\boldsymbol{x}, t) = \sum_{i,j,k=0}^{\infty} \int_{-\infty}^{\infty} \frac{(-1)^{i+j+k}}{i!j!k!} \frac{\partial^{i+j+k} G_0(\tau - t, r_0)}{\partial x_1^i \partial x_2^j \partial x_3^k} \int_{V_S} (\xi_1 - \xi_1^0)^i (\xi_2 - \xi_2^0)^j (\xi_3 - \xi_3^0)^k Q_p(\boldsymbol{\xi}, \tau)\, dV(\xi)\, d\tau$$

where it is emphasized that $r_0 = |\boldsymbol{x} - \boldsymbol{\xi}_0|$ is not a function of $\boldsymbol{\xi}$ anymore. We may now use the actual form of the free field Green's function $G_0 = \delta(\tau - t + r_0/a_\infty)/(4\pi r_0)$ at $\boldsymbol{\xi}_0$ in order to evaluate the time integration and we finally obtain the multipole expansion w.r.t. G_0 of the source Q_p:

$$p'(\boldsymbol{x}, t) = \sum_{i,j,k=0}^{\infty} (-1)^{i+j+k} \frac{\partial^{i+j+k}}{\partial x_1^i \partial x_2^j \partial x_3^k} \left[\frac{m_{ijk}(\tau_0)}{4\pi r_0}\right] \tag{144}$$

$$m_{ijk}(\tau_0) = \int_{V_S} \frac{1}{i!j!k!} (\xi_1 - \xi_1^0)^i (\xi_2 - \xi_2^0)^j (\xi_3 - \xi_3^0)^k\, Q_p(\boldsymbol{\xi}, \tau_0)\, dV(\xi) \tag{145}$$

where $\tau_0 := t - r_0/a_\infty$ is the retarded time with respect to the (fixed) reference point $\boldsymbol{\xi}_0$ therefore being independent of $\boldsymbol{\xi}$ and again $r_0 := |\boldsymbol{x} - \boldsymbol{\xi}_0|$. The abreviation m_{ijk} is called "multipole moment of order 2^{i+j+k}" of the source Q_p. The zeroth term m_{000} is called "monopole moment", while all first terms m_{100}, m_{010} and m_{001} are called "dipole moments". The next terms are then termed "quadrupole moment" $(i + j + k = 2)$ and "octupole moment" $(i + j + k = 3)$ etc.

For very large distances r_0 of the observer to the source the multipole expansion attains a more simple form (terms proportional to second or higher power of r_0^{-1} neglected):

$$p'(\boldsymbol{x}, t) \simeq \frac{1}{4\pi r_0} \sum_{i,j,k=0}^{\infty} a_\infty^{-(i+j+k)} \underbrace{\left(\frac{\partial r_0}{\partial x_1}\right)^i \left(\frac{\partial r_0}{\partial x_2}\right)^j \left(\frac{\partial r_0}{\partial x_3}\right)^k}_{\cos^i \vartheta_1\ \cos^j \vartheta_2\ \cos^k \vartheta_3} \frac{\partial^{i+j+k} m_{ijk}(\tau_0)}{\partial t^{i+j+k}} \tag{146}$$

Note that the derivatives of r_0 with respect to the coordinates represent nothing but direction cosines $\cos \vartheta_l$ relative to the coordinate axes x_l, i.e. for higher multipole moments we have a spatially very complex "hedgehog" pattern, consisting of so called *radiation lobes* (German: "Abstrahlungskeulen"). Figure *18* depicts examples of various combinations of i, j, k. The diagrams only show the $x_1 - x_2$ section plane through the 3D characteristics; the actual monopole characteristic is a sphere, while the dipoles represent two spheres each. According to its definition in eqn. (146), the $x_1 - x_2$-quadrupole may be generated by multiplication of the two x_1 and

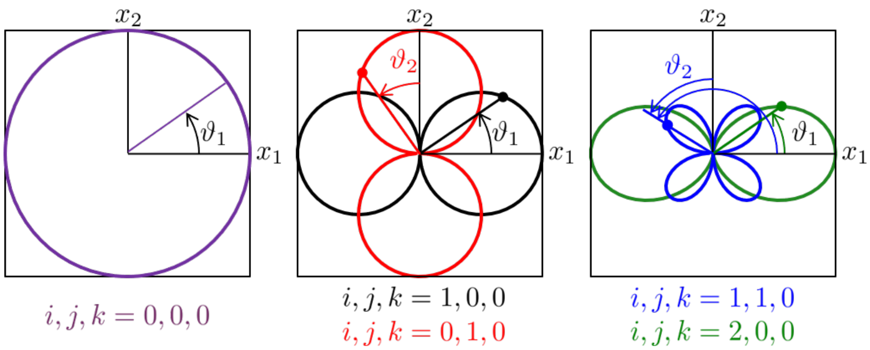

Figure 18: Radiation characterisitcs of various components i, j, k of the multipole expansion; left: monopole, center: x_1- and x_2-dipoles, right: x_1-longitudinal and $x_1 - x_2$-quadrupole.

x_2-dipoles. Although the longitudinal quadrupole seems similar to a dipole, it is fundamentally different not only because of its ellipsoidal instead of spherical lobes, but by the fact that both lobes represent the same phase while a diplole features oppositely phased lobes. Note also, that only one term of the infinite sum remains for the case of an incompressible medium ($a_\infty \to \infty$), namely the monopole part.

2.6.4.1 Expansion of source term Q_p of (61)

Let us again take a look at the form of the source $Q_p =: Q_{m\vartheta} + Q_f$ from (61). It is composed of the time derivative of a scalar and the divergence of a vector with

$$Q_p = \overbrace{\frac{\partial}{\partial t} \underbrace{\left\{ \left(1 - \frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty}\right) \dot{m}' + \frac{\sigma_\infty}{T_\infty} \dot{\vartheta}' \right\}}_{=:\, \dot{\theta}'}}^{=:\, Q_{m\vartheta}} \overbrace{- \boldsymbol{\nabla} \cdot \boldsymbol{f}'}^{=:\, Q_f} \tag{147}$$

In a first step we determine the monopole contribution of Q_p to the sound field p'. According to (145) the monopole moment of $Q_{m\vartheta}$ is

$$m_{000}[Q_{m\vartheta}] = \int\limits_{V_S} Q_{m\vartheta}\, dV(\xi) = \frac{\partial}{\partial t} \int\limits_{V_S} \dot{\theta}'\, dV(\xi)$$

This represents the time change of the bulk mass and heat flow over the whole source volume. The monopole moment of $Q_f := -\boldsymbol{\nabla} \cdot \boldsymbol{f}'$ is correspondingly

$$m_{000}[Q_f] = \int\limits_{V_S} Q_f\, dV(\xi) = -\int\limits_{V_S} \boldsymbol{\nabla}_\xi \cdot \boldsymbol{f}'\, dV(\xi) = -\int\limits_{\partial V_S} \boldsymbol{f}' \cdot \boldsymbol{n}\, dS = 0$$

The last equality follows because the source region is a finite volume V_S. There is no contribution from the external forces to the monopole moment of Q_p. The overall monopole contribution of Q_p to the pressure field p' is therefore (144)

$$p'_m := p'_{000} = \frac{1}{4\pi r_0}\frac{\partial}{\partial t}\int\limits_{V_S} \dot\theta'(\xi,\tau_0)\, dV(\xi) \tag{148}$$

with $r_0 = |\boldsymbol{x} - \boldsymbol{\xi}_0|$ being the distance of the observer to the source and $\tau_0 := t - r_0/a_\infty$ the respective retarded time (of course both depending on the choice of the reference point $\boldsymbol{\xi}_0$ in V_S). We summarize: The monopole moment is determined by the bulk flow of mass and heat, while all other terms (due to $\boldsymbol{f}'$) vanish. Conversely one characterizes the effect of applied fluctuating mass and heat sources as of monopole type (although this really only concerns Q_p's spatial mean, i.e. the source strength).

The first dipole term of Q_p is evaluated similarly as above yielding

$$m_{100}[Q_{m\vartheta}] = \frac{\partial}{\partial t}\int\limits_{V_S}(\xi_1 - \xi_1^0)\dot\theta'\, dV(\xi) \quad \text{and} \quad m_{100}[Q_f] = \int\limits_{V_S} f'_1\, dV(\xi)$$

We observe, that the dipole moment $m_{100}[Q_f]$ of the force contribution Q_f represents the first component of the bulk force vector exerted by the source volume on the fluid. The second/third terms (010)/(001) follow analogously such that the overall dipole contribution of Q_p to the pressure field p' according to the definition (144) is

$$p'_d := p'_{100}+p'_{010}+p'_{001}=\boldsymbol{\nabla}_x\boldsymbol{\cdot}\left\{\frac{-1}{4\pi r_0}\left[\frac{\partial}{\partial t}\int\limits_{V_S}(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\,\dot\theta'(\boldsymbol{\xi},\tau_0)\, dV(\xi)+\int\limits_{V_S}\boldsymbol{f}'(\xi,\tau_0)\, dV(\xi)\right]\right\} \tag{149}$$

Note that by Gauss' theorem we could have written the first integral equivalently as $-\int_{V_S}(\boldsymbol{\xi} - \boldsymbol{\xi}_0)(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\boldsymbol{\cdot}\boldsymbol{\nabla}_\xi\, \dot\theta'(\boldsymbol{\xi},\tau_0)\, dV(\boldsymbol{\xi})$ stating explicitly, that it expresses the effects of a spatial variation of $q'_{m\vartheta}$ over the source region. We summarize: The dipole moment is determined by the mean force vector and the mean variation of the heat/mass flux over the source region. Apart from such spatial variation effects the dipole contribution of the source is generically determined by the force term. Conversely one characterizes the effect of an applied fluctuating force $\boldsymbol{f}'$ as of dipole type (although this really only concerns $\boldsymbol{f}'$'s spatial mean, the bulk force applied).

Let us finally write down the solution p' approximated up to the quadrupole contribution $p'_2(\boldsymbol{x},t) := p'_m + p'_d + p'_q$, and compare with the form (105) of the general solution for $p'(\boldsymbol{x},t) \simeq p'_2(\boldsymbol{x},t)$

$$\begin{aligned} p'_2(\boldsymbol{x},t) &= \frac{1}{4\pi}\left\{\frac{\partial}{\partial t}\int\limits_{V_S}\frac{1}{r_0}\left[\left(1-\frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty}\right)\dot m' + \frac{\sigma_\infty}{T_\infty}\dot\vartheta'\right]_{\tau_0} dV(\xi) - \boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{V_S}\frac{\boldsymbol{f}'(\tau_0,\boldsymbol{\xi})}{r_0}\, dV(\xi)\right\} \\ &\quad -\boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{V_S}\frac{(\boldsymbol{\xi}-\boldsymbol{\xi}_0)}{4\pi r_0}\,\frac{\partial\dot\theta'(\boldsymbol{\xi},\tau_0)}{\partial t}\, dV(\xi) \\ &\quad +\boldsymbol{\nabla}_x\boldsymbol{\cdot}\boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{V_S}\frac{(\boldsymbol{\xi}-\boldsymbol{\xi}_0)}{4\pi r_0}\left\{(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\frac{\partial\dot\theta'(\boldsymbol{\xi},\tau_0)}{\partial t} - \boldsymbol{f}'(\tau_0,\xi)\right\} - \frac{\boldsymbol{f}'(\tau_0,\boldsymbol{\xi})(\boldsymbol{\xi}-\boldsymbol{\xi}_0)}{4\pi r_0}\, dV(\xi) \end{aligned} \tag{150}$$

The comparison with the terms of (105) shows two differences:

i) although the first line of (150) resembles in shape equation (105) the integrals are evaluated at the retarded time $\tau_0 = t - |\boldsymbol{x} - \boldsymbol{\xi}_0|/a_\infty$ instead of $\tau = t - |\boldsymbol{x} - \boldsymbol{\xi}|/a_\infty$, i.e. variations of retarded time over the source domain are neglected,

ii) there are extra terms on the second and third line of (150). The integral on the second line represents an additional dipole contribution and is the first order correction to the above mentioned neglect in retarded time variations of the mass/heat source over V_S when reducing the mass/heat source integral to a monopole. The integral on the third line is a quadrupole term. Its first part is the second order correction to neglecting retarded time variations of the mass/heat source. The second part of the last integral is the first order correction to the neglect of retarded time variations on the forcing made when reducing the force integral to a dipole.

This shows that the neglect of retarded time variations over the source generates higher order terms, even though the "main characteristics" of the mass/heat source is captured by the monopole term and the "main characteristics" of the force is captured by the dipole term. Note that the magnitude of all these higher order terms are through $(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$ or its multiples directly proportional to the size of the domain. Therefore we may conclude that the multipole expansion series will converge faster the smaller the source domain.

2.6.4.2 Point sources. The abstraction of a point source is motivated by the multipole expansion. Higher order expansion terms become smaller with the size of the domain. Without having the difficulty of defining what we mean when distinguishing between "large" and "small" sources we may consider the limiting case of sources of infinitesimally small source domain V_S, located at $\boldsymbol{\xi}_0$ with infinitely concentrated source strengths. Such "point sources" may be expressed explicitly using the Delta function:

$$\begin{aligned} \dot{m}'(\boldsymbol{x},t) &= \dot{m}'_p(t)\,\delta(\boldsymbol{x}-\boldsymbol{\xi}_0) \\ \dot{\vartheta}'(\boldsymbol{x},t) &= \dot{\vartheta}'_p(t)\,\delta(\boldsymbol{x}-\boldsymbol{\xi}_0) \\ \boldsymbol{f}'(\boldsymbol{x},t) &= \boldsymbol{f}'_p(t)\,\delta(\boldsymbol{x}-\boldsymbol{\xi}_0) \end{aligned} \tag{151}$$

If we insert this into (150) all the higher order correction terms vanish identically such that the expansion up to the shown order is exact

$$p'(\boldsymbol{x},t) = p'_2 = p'_1 = \frac{1}{4\pi}\left\{\frac{1}{r_0}\underbrace{\left[\left(1-\frac{\sigma_\infty p_\infty}{\rho_\infty T_\infty}\right)\frac{\partial \dot{m}'_p}{\partial t}\Big|_{\tau_0} + \frac{\sigma_\infty}{T_\infty}\frac{\partial \dot{\vartheta}'_p}{\partial t}\Big|_{\tau_0}\right]}_{=(Q_{m\vartheta})_p} - \boldsymbol{\nabla}_x\cdot\left[\frac{\boldsymbol{f}'_p}{r_0}\right]_{\tau_0}\right\} \tag{152}$$

If the mass and heat sources are point sources they represent pure monopoles and a point force appears as a pure dipole. The source strength of the point monopoles corresponds to the (concentrated) monopole moment, while the point source $\boldsymbol{f}'_p$ represents the (concentrated) dipole moment. When comparing the monopole part of the solution with our elementary, spherically symmetric solution (95) we see that the latter was nothing but a monopole point source. Moreover the comparison of monopole and dipole terms in (152) shows that a dipole field formally appears to be the spatial derivative of monopole fields with different source strengths for each direction.

2.6.4.3 A line source of finite extend. Some sources of noise are not appropriately modeled by assuming that their spatial extension is negligible as in (2.6.4.2). Let us consider for this purpose the sound radiation from a circular cylinder of diameter d and length l in a cross flow of speed U_∞. Known as von Karman vortex street, the flow past a cylinder exibits a typcial transversal oscillation of its wake, with a characteristic Strouhal number (dimensionless frequency f) $Sr = fd/U_\infty$ of $Sr \approx 0.2$. The wake's oscillation is connected with an oscillation of lift about the value zero. Conversely, the cylinder exerts an equivalent force on the fluid. At each spanwise location we represent the action of the cylinder on the fluid by a line force of length l pointing perpendicular to flow and cylinder axis. The section lift of this line force may then be expressed like:

$$\boldsymbol{f}' = \boldsymbol{e}_2 F(\tau;\xi_3)\delta(\xi_1)\delta(\xi_2)H(l/2-\xi_3)H(l/2+\xi_3)$$

where the origin of the coordinate system is located in the symmetry point of the cylinder, while the ξ_1 and ξ_3 directions point along the free stream and the span respectively. For simplicity of the presentation from now on we assume that the section lift is constant along the span of the cylinder, i.e. $F = F(\tau) \neq F(\tau, \xi_3)$.

We solve for the far field of the sound radiated by the cylinder by applying the multipole expansion according to (146) with the source term being $Q_p = \boldsymbol{\nabla}\cdot\boldsymbol{f}'$. Since $\boldsymbol{f}'$ contains the ξ_2 component only, $Q_p = \frac{\partial f_2'}{\partial \xi_2}$. Inserted into the expression for the multipole moment (145) and integrating by parts we have

$$\begin{aligned} m_{ijk} &= -\frac{F(\tau_0)}{i!j!k!}\int\limits_{V_S} \frac{\partial \xi_1^i \xi_2^j \xi_3^k}{\partial \xi_2}\,\delta(\xi_1)\delta(\xi_2)H(l/2-\xi_3)H(l/2+\xi_3)\,d\xi_1 d\xi_2 d\xi_3 \\ &= F(\tau_0)\frac{1}{k+1!}(\frac{l}{2})^{k+1}[(-1)^{k+1}-1]\,\delta_{0i}\,\delta_{1j} \end{aligned}$$

were δ_{kl} denotes Kronecker"s symbol, assuming unity whenever $k = l$ and the value zero for $k \neq l$. Insertion into (146) gives

$$\begin{aligned} p'(\boldsymbol{x},t) &\simeq \frac{1}{4\pi r_0}\sum_{k=0}^{\infty}\frac{1}{k+1!}[(-1)^{k+1}-1]\,a_\infty^{-(k+1)}\,\cos\vartheta_2\,\cos^k\vartheta_3\,(\frac{l}{2})^{k+1}\,\frac{d^{k+1}F(\tau_0)}{d\tau_0^{k+1}} \\ &= -\frac{2}{4\pi r_0}\frac{\cos\vartheta_2}{\cos\vartheta_3}\left\{(\frac{l\cos\vartheta_3}{2a_\infty})F' + \frac{1}{6}(\frac{l\cos\vartheta_3}{2a_\infty})^3 F''' + \frac{1}{120}(\frac{l\cos\vartheta_3}{2a_\infty})^5 F^V + \dots\right\} \end{aligned}$$

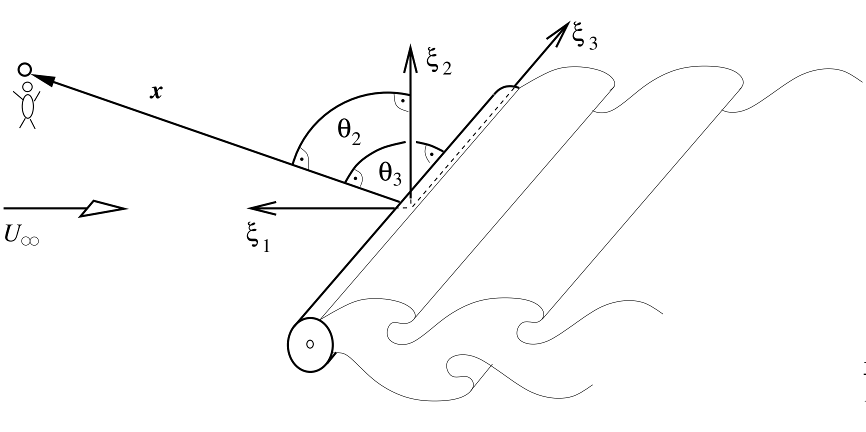

Figure 19: Cylinder in cross flow.

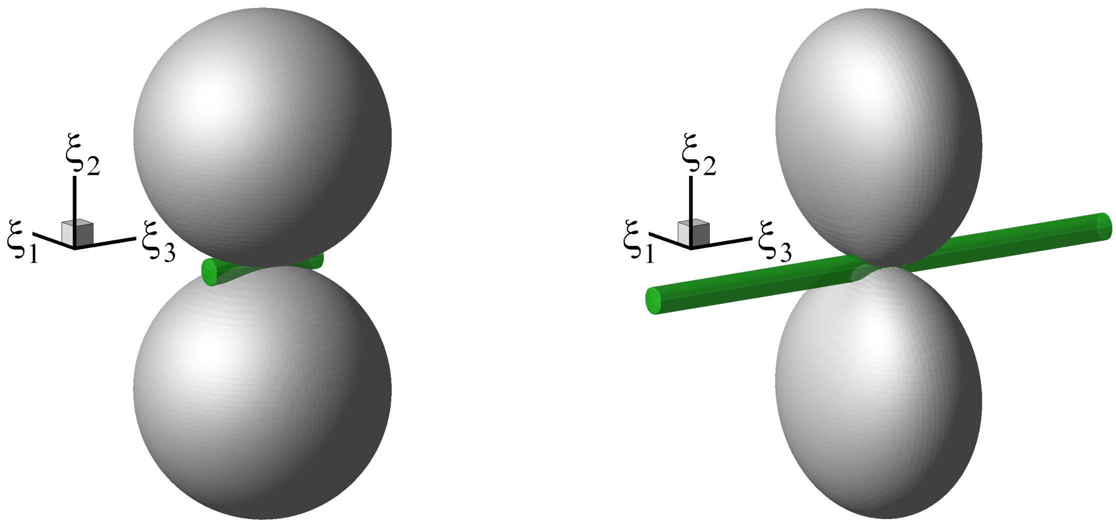

Figure 20: Directivity $\tilde{D}$ of sound radiation from cylinder in cross flow. The Strouhal number is assumed to be $Sr = 0.2$; left: $M = 0.1$, $l/d = 5$; right: $M = 0.2, l/d = 20$.

where ϑ_2 (ϑ_3) is the angle between the ξ_2 (ξ_3) direction and the direction defined by symmetry point of the cylinder and observer location. In order to make the solution even more explicit let us assume that the oscillation is purely sinusoidal and that the rms-value of the section lift coefficient $\tilde{c}_l$ is near 0.3 or so. Then

$$F(\tau_0) = \sqrt{2}\,\tilde{c}_l\, d\, \tfrac{1}{2}\rho_\infty U_\infty^2\, \sin(\omega t - kr_0)$$

with $\omega = Sr 2\pi U_\infty/d$ and the wave number $k = 2\pi Sr M/d$. For this sinusoidal oscillation we find the series in the expression for p' to represent the sine, i.e.

$$p'(\boldsymbol{x}, t) = -\frac{\sqrt{2}}{4\pi r_0}\,\tilde{c}_l\,\rho_\infty U_\infty^2\, d\, \underbrace{\frac{\cos\vartheta_2}{\cos\vartheta_3}\, \sin(\pi\, Sr\, M\, \cos\vartheta_3\, l/d)}_{:= D} \cos(\omega t - kr_0)$$

The dependence of the radiation on the direction is represented in the so called directivity D; surely, the cylinder does not radiate the same in all directions. The normalized directivity $\tilde{D} := D/D_{\max}$ is depicted in figure *20* for two different length-to-depth ratios l/d and flow Mach numbers. For small arguments of the sine (like the case shown for $l/d = 5$, $M = 0.1$ in fig.*20*), it may be replaced by the argument itself. Then $\tilde{D} \longrightarrow \cos\vartheta_2$ and the directivity becomes dipole-like with the lobes pointing in the direction of the lift. For larger arguments of the sine, the directivity becomes more flat in the lateral direction.

We may also determine the sound power radiated by the cylinder. From (83) we know that the intensity in the far field is $I = \overline{p'^2}/(\rho_\infty a_\infty)$ which we have to integrate according to (12) over a closed surface, chosen to be a sphere of radius r_0. In order to do so, we choose a polar coordinate system with $\xi_1 = r_0 \cos\varphi \sin\vartheta_3$, $\xi_2 = r_0 \sin\varphi \sin\vartheta_3$, $\xi_3 = r_0 \cos\vartheta_3$, in which the surface element is $dS = r_0^2 \sin\vartheta_3 \, d\varphi \, d\vartheta_3$.

$$P = \int_0^{2\pi}\int_0^{\pi} \frac{\overline{p'^2}}{\rho_\infty a_\infty} r_0^2 \sin\vartheta_3 \, d\vartheta_3 \, d\varphi = \frac{2r_0^2}{\rho_\infty a_\infty}\int_0^{2\pi}\int_0^{\pi/2} \overline{p'^2} \sin\vartheta_3 \, d\vartheta_3 \, d\varphi$$

where we used the symmetry about $\vartheta_3 = \pi/2$. In order to insert our solution for p' we still have to express the direction cosine $\cos\vartheta_2$ in these coordinates. Since $\xi_2 = r_0 \cos\vartheta_2$ we obtain $\cos\vartheta_2 = \sin\varphi \sin\vartheta_3$ and we get

$$\begin{aligned} P &= \frac{\rho_\infty \tilde{c}_l^2 U_\infty^3 M d^2}{8\pi^2} \overbrace{\int_0^{2\pi} \sin^2\varphi \, d\varphi}^{\pi} \int_0^{\pi/2} \tan^2\vartheta_3 \sin^2(\pi \, Sr \, M \, \cos\vartheta_3 \, l/d) \sin\vartheta_3 \, d\vartheta_3 \\ &= \frac{\rho_\infty \tilde{c}_l^2 U_\infty^3 M d^2}{8\pi}\Big[\frac{1}{2}\Big(\frac{\sin(2\alpha)}{2\alpha} - 1\Big) - \sin^2\alpha + \alpha \, \mathrm{si}(2\alpha)\Big] \end{aligned}$$

where $\alpha := \pi \, Sr \, M \, l/d$ and $\mathrm{si}(x) := \int_0^x (\sin(t)/t)dt$ is called "sine integral" (german: "Integralsinus"), which is a tabulated function. For small arguments of $\alpha \to 0$, say, for small Mach number or small l/d this expression can be simplified by a Taylor expansion about $\alpha = 0$ (leaving $\frac{2}{3}\alpha^2$ for the square bracket):

$$P = \frac{\pi}{12}\tilde{c}_l^2 \rho_\infty U_\infty^3 M^3 Sr^2 l^2 \qquad (153)$$

For very large α the last term in the square brackets dominates, leaving $\frac{\pi}{2}\alpha$ since $\mathrm{si}(\infty) = \pi/2$ and we have

$$P = \frac{\pi}{16}\tilde{c}_l^2 \rho_\infty U_\infty^3 M^2 Sr \, l \, d \qquad (154)$$

The result is shown in the left diagram of Fig *21* for $M = 0.3$. For small l/d we see that the general solution asymptotes to (153) and for large α to (154). Note that flow effects in the propagation of the sound waves have been neglected so far. This may be justified for small Mach numbers. The Mach number dependence is shown in the right diagram of Fig *21*.

According to (153), i.e. when α is small, the sound power scales like the sixth power of the speed, which is a typical result for so called "compact" objects (see next paragraph). Note that for this case the source could have been approximated by a point force in the center of the cylinder. Let us estimate the critical l/d, up to which the cylinder behaves like a compact source. We simply determine the intersection of the two asymptotics (153) and (154) being at $l/d = \frac{3}{4SrM}$. We may express SrM by $SrM = He \, d/l$, where $He = l/\lambda$ is called "Helmholtz number", which may be interpreted as the ratio between the characteristic dimension of the object in relation to the radiated wavelength. We may therefore say that the cylinder behaves as a compact source as long as $He < 3/4$, i.e. whenever the cylinder is clearly shorter than one wavelength. This result is very general and not only restricted to cylinders.

For considerably larger l/d the speed scaling exponent of the sound power reduces to five, a reduction of one in the exponent of which is a very typical result for a 2D source compared to the 3D case. The reason for the reduced increase of the sound power with the cylinder length or Mach number in this regime is due to wave cancellations. However, for realistic flows the wake oscillation is not perfectly correlated in the spanwise direction as we assumed. Typical spanwise correlation lengths are even less than 5 diameters or so. For elastically supported cylinders though, the oscillation may trigger a coherent vortex shedding along the whole cylinder length. Then very loud tones may occur (e.g. electric transmission lines in wind).

2.6.5 Compactness, nearfield and farfield

In this section we will show that the character of the pressure field $\hat{p}(\boldsymbol{x}, \omega)$ changes depending on whether the point of observation is near or far away from the source. The distinction between the different regions in the pressure field is important because it may help us in the actual solution of an acoustic problem. Even for the most simple applied problems it usually is impossible to obtain closed-form solutions. Then it becomes necessary to introduce simplifications and the following distinction between near- and farfield as well as the notion of compactness should give us a valuable and rather general guideline for simplifications.

2.6.5.1 Compactness In the characterization of sources and acoustic fields it is useful to introduce the characteristic dimensionless parameter $He := l/\lambda$, which is called "Helmholtz number" and represents the ratio of a geometrical length scale l of an object (source or body) to an acoustic length λ (wavelength). The object is called "acoustically compact" when $He \ll 1$. Otherwise the object is called "acoustically non-compact". Note that this distinction depends on the wavelength considered, and thus the corresponding frequency $\omega = 2\pi a_\infty/\lambda$. For practical purposes $He \lesssim 1/4$ may still be a reasonable range for compactness. Introducing the characteristic

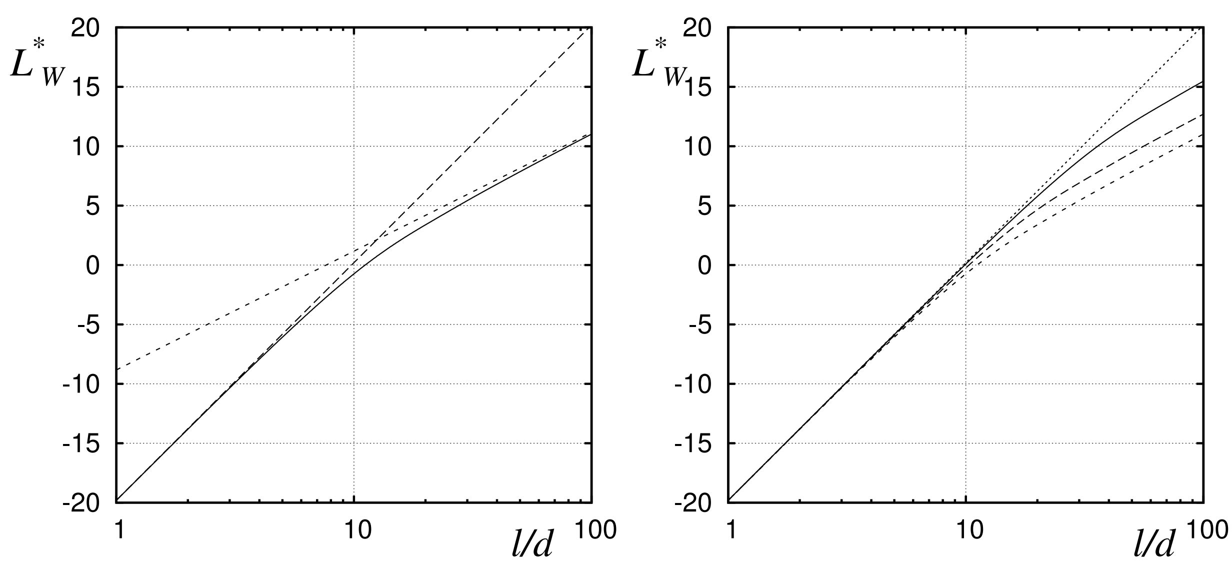

Figure 21: Normalized sound power level $L^*_W = 10\,\mathrm{lg}[P/(\tilde{c}_l^2\rho_\infty d^2 M^6 a_\infty^3)]$ as a function of dimensionless length l/d; left: $M = 0.3$ and asymptotics for $l/d \to 0$ according to (153) and for large l/d; right: solid curve - $M = 0.1$, long-dashed curve - $M = 0.2$, dashed curve - $M = 0.3$, dotted curve - asymptotic (153).

lengths for the source and body sizes as in figure *22*

i) the maximum linear extension of the source region $l_S = \max_{\xi,\xi_S \in \partial V_S} [|\boldsymbol{\xi} - \boldsymbol{\xi}_S|]$ and

ii) the maximum linear extension of the body $l_B = \max_{\xi,\xi_S \in \partial V_B} [|\boldsymbol{\xi} - \boldsymbol{\xi}_S|]$.

we may characterize the relative geometrical size of our source by $He_S = l_S/\lambda$. Whether the body is acoustically compact or not is a matter of the value of $He_B = l_B/\lambda$.

As a reference to the preceeding section we may now say that multipole expansions converge fast for compact sources only. The limiting case of a compact source is a point source with $He_S = 0$.

2.6.5.2 Nearfield and farfield The existence of subdomains of different character in the pressure field was already seen in the formulation (104) of the general free field solution, which explicitly showed that a certain part of the scaled pressure signal $(\hat{p} \cdot r)$ due to $\boldsymbol{f}'$ (dipole part of source) survives as the distance from the source r becomes asymptotically large, while another part dies out for $r \to \infty$, in turn being dominant close to the source. Moreover we saw that the part, surviving for large r was compressible (and unsteady) in nature, since for an incompressible fluid it would vanish identically.

In order to decide, whether the pressure field at a point $\boldsymbol{x}$ is of compressible or incompressible nature the general solution (131) is considered. The solution contains the special case of the pressure field of an incompressible fluid given the same sources. The compressibility effect enters entirely through the wave number $k = \omega/a_\infty$ and we may simply let $k \to 0$ in order to arrive at the incompressible solution. We note however, that k always appears in conjunction with r as dimensionless expression $kr = \omega r/a_\infty$. Obviously for an incompressible fluid $kr = 0$ and consequently those points $\boldsymbol{x}$ for which $kr \ll 1$ will have a pressure field like of an incompressible fluid, which is usually much more simple to describe.

There are two distinguished effects on the solution (131) due to finite kr: i) the phase factor $\exp(ikr)$, which represents the retarded time and ii) the term proprotional to ikr in the surface integral. Given an observation point $\boldsymbol{x}$ we have to evaluate the size of $kr = k|\boldsymbol{x} - \boldsymbol{\xi}|$ as $\boldsymbol{\xi}$ sweeps over the entire source region V_S and the entire surface ∂V_B of the body V_B. It is convenient to

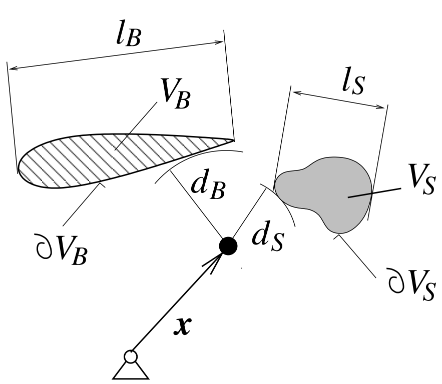

Figure 22: Characteristic length scales for source and body

define a dimensionless parameter, $\tilde{\Pi}$ which describes the character of the pressure field at $\boldsymbol{x}$:

$$\tilde{\Pi} := \max_{\xi \in \partial V_S, \partial V_B} \underbrace{|\boldsymbol{x} - \boldsymbol{\xi}|/\lambda}_{= kr/2\pi} \tag{155}$$

where $\lambda = 2\pi/k$ is the wave length of an acoustic wave of radian frequency $\omega = ka_\infty$.

We may more conveniently determine an upper estimate Π of $\tilde{\Pi}$ when introducing the following two additional characteristic length scales: (see fig. *22*)

iii) the distance $d_S(\boldsymbol{x}) = \min_{\xi \in \partial V_S} [|\boldsymbol{x} - \boldsymbol{\xi}|]$ of the observer to the source region,

iv) the distance $d_B(\boldsymbol{x}) = \min_{\xi \in \partial V_B} [|\boldsymbol{x} - \boldsymbol{\xi}|]$ of the observer to the body (with the convention $d_B = 0$ when no body present),

The dimensionless parameter

$$\Pi(\boldsymbol{x}) := \frac{\max[d_S(\boldsymbol{x}) + l_S, d_B(\boldsymbol{x}) + l_B]}{\lambda} = \max\left[\frac{d_S(\boldsymbol{x})}{\lambda} + He_S, \frac{d_B(\boldsymbol{x})}{\lambda} + He_B\right] \tag{156}$$

enables us to divide the pressure field into respective subdomains.

For $\Pi \ll 1$ the pressure is the same as the pressure in an unsteadily moving incompressible medium, i.e. it is of purely hydrodynamic nature and we would have obtained the same result had we considered the incompressible conservation equations right away. Note also that the solution becomes independent of the speed of sound. We call the domain where $\Pi(\boldsymbol{x}) \ll 1$ the *acoustic nearfield* (German: "akustisches Nahfeld") of the pressure. It is -as mentioned- characterized by incompressible phenomena.

Let us first assume no bodies present, i.e. a free field problem $d_B = 0$, $He_B = 0$. Then, according to (156), conditions of an acoustic nearfield are fulfilled only in cases when the acoustic wavelength is large compared to the observer's distance from the source $d_S(\boldsymbol{x})$ and the characteristic extension of the source l_S (compact source). As either the distance from the source or the extension of the source increases, so does the unsteady compressibility effect and (at least part of) the pressure attains acoustic nature.

We may think of cases, where there exists no true acoustic nearfield, because from (156) we see that even for very near the source $d_S \to 0$ it may happen, that the parameter $\Pi \not\ll 1$, namely when the source is non-compact. The region where the distance $d_S \ll \lambda$, while $l_S/\lambda \not\ll 1$ is then sometimes called *geometric nearfield* (German: "geometisches Nahfeld"). In this region one may expect the solution to be influenced both by hydrodynamic and acoustic processes. Note, the distinction between acoustic and geometric nearfield makes sense for non-compact sources only.

The region, which is dominated by unsteady, compressible processes is characterized by $d_S(\boldsymbol{x})/\lambda \not\ll 1$, no matter the characteristic geometric extension of the source l_S. This region is called *acoustic farfield* (German: "akustisches Fernfeld").

Yet another region can be identified, namely the so-called *geometric farfield* (German: "geometrisches Fernfeld"), which is the part of the acoustic farfield, for which $l_S \ll d_S(\boldsymbol{x})$. The distinction between acoustic and geometric farfield makes sense for non-compact sources.

The presence of a body V_B is taken into account when the general definition of Π from (156) is used. For instance a point lying in the acoustic near field of a source becomes part of the farfield as soon as a body is placed far enough from the observer or the source. The reason is, that sound emanating from the source interacts with the body, becomes partially reflected (or diffracted) and reaches the observer point, at which the pressure is therefore composed not only of the direct (incompressible) part of the source, but as well contains these acoustic components. The same is true for a non-compact body no matter what geometric distances.

3 Moving medium / moving source

It is very difficult to describe the propagation of sound in a general flow field uniquely. We therefore consider certain classes of flows, for which we may derive respective (generalized) acoustic wave equations.

3.1 Sound propagation in steady parallel flows

Let us consider a steady mean flow field

$$\boldsymbol{v}^0(\boldsymbol{x}) = \boldsymbol{e}_x u^0 \tag{157}$$

which (even for compressible medium) implies (see appendix D), that $u^0 \neq u^0(x)$, i.e.

$$u^0 = u^0(y,z) = u^0(h(y,z)) \tag{158}$$

allowing for variation along the normal directions y, z. Constant values of the generalized normal coordinate h represent the iso-surfaces, on which the flow is constant (see figure 23). For instance in the case of axisymmetric parallel free jet flow $h(y,z) = \sqrt{y^2+z^2}$ is the distance to the jet axis. In the most simple case $h(y,z) = z$, which represents a plane shear layer. First we need to check, what the accompanying pressure and the density field may look like such that $\rho^0, \boldsymbol{v}^0, p^0$ form a true flow field. We insert $\boldsymbol{v}^0$ into the mass balance equation (51) for the steady flow and obtain

$$\boldsymbol{v}^0 \cdot \boldsymbol{\nabla} \rho^0 + \rho^0 \underbrace{\boldsymbol{\nabla}\cdot\boldsymbol{v}^0}_{= \frac{\partial u^0}{\partial x} = 0} = 0$$

which tells us, that the gradient of the mean density field ρ^0 needs to be orthogonal to the flow direction. The variation $\rho^0(h)$ may be any function. Let us analogously check the satisfaction of the momentum equation (52)

$$\rho^0 \underbrace{\boldsymbol{v}^0\cdot\boldsymbol{\nabla}\boldsymbol{v}^0}_{=0} + \boldsymbol{\nabla} p^0 = \boldsymbol{0} \Longrightarrow p^0 = const = p_\infty$$

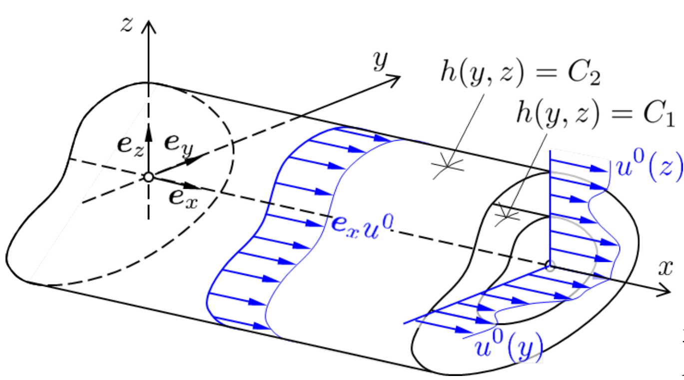

Figure 23: Co-ordinates for description of parallel flow.

As we did for the medium at rest we try to derive one single equation for the pressure perturbation p'. First the perturbation equations need to be expressed for the special case of parallel flow. The momentum equation (55) reads:

$$\frac{D^0\boldsymbol{v}'}{Dt} + \frac{1}{\rho^0}\boldsymbol{\nabla} p' + \underbrace{\boldsymbol{v}'\cdot\boldsymbol{\nabla}\boldsymbol{v}^0}_{\left(\underbrace{v'\frac{\partial u^0}{\partial y}+w'\frac{\partial u^0}{\partial z}}_{\underbrace{v'\frac{du^0}{dh}\frac{\partial h}{\partial y}+w'\frac{du^0}{dh}\frac{\partial h}{\partial z}}_{\frac{du^0}{dh}\boldsymbol{v}'\cdot\underbrace{\frac{\boldsymbol{\nabla} h}{|\boldsymbol{\nabla} h|}}_{\boldsymbol{e}_h}|\boldsymbol{\nabla} h| = \frac{du^0}{dh}v'_h|\boldsymbol{\nabla} h|}}\right)\boldsymbol{e}_x} + \overbrace{\rho'\boldsymbol{v}^0\cdot\boldsymbol{\nabla}\boldsymbol{v}^0}^{=0} = \frac{1}{\rho^0}\boldsymbol{f}' \tag{159}$$

where $\frac{D^0}{Dt} := \frac{\partial}{\partial t} + u^0\frac{\partial}{\partial x}$. Next we write down the pressure equation (56):

$$\frac{1}{a_0^2}\frac{D^0 p'}{Dt} + \rho^0\boldsymbol{\nabla}\cdot\boldsymbol{v}' = \dot{\theta}'$$

Note that the density perturbation ρ' has dropped out of these two equations for p' and $\boldsymbol{v}'$. In order to eliminate the velocity $\boldsymbol{v}'$ we take the divergence of the above momentum equation

$$\frac{\partial\boldsymbol{\nabla}\cdot\boldsymbol{v}'}{\partial t} + \underbrace{\boldsymbol{\nabla}\cdot\left(u^0\frac{\partial\boldsymbol{v}'}{\partial x}\right)}_{u^0\frac{\partial}{\partial x}(\boldsymbol{\nabla}\cdot\boldsymbol{v}') + \frac{\partial\boldsymbol{v}'}{\partial x}\cdot\underbrace{\boldsymbol{\nabla} u^0}_{\frac{du^0}{dh}\frac{\boldsymbol{\nabla} h}{|\boldsymbol{\nabla} h|}|\boldsymbol{\nabla} h|}} + \frac{du^0}{dh}|\boldsymbol{\nabla} h|\frac{\partial v'_h}{\partial x} + \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{\nabla} p'\right) = \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{f}'\right)$$

which in summary is

$$\frac{D^0\boldsymbol{\nabla}\cdot\boldsymbol{v}'}{Dt} + 2|\boldsymbol{\nabla} h|\frac{du^0}{dh}\frac{\partial v'_h}{\partial x} + \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{\nabla} p'\right) = \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{f}'\right)$$

If we now take the material derivative $\frac{D^0}{Dt}$ of the above pressure equation, we generate again the expression $\frac{D^0\boldsymbol{\nabla}\cdot\boldsymbol{v}'}{Dt}$ which may be eliminated:

$$\frac{1}{a_0^2}\frac{D^{02}p'}{Dt^2} + \rho^0\frac{D^0\boldsymbol{\nabla}\cdot\boldsymbol{v}'}{Dt} = \frac{D^0\dot{\theta}'}{Dt}$$

The combination of the last two equations finally gives:

$$\frac{1}{a_0^2}\frac{D^{02}p'}{Dt^2} - \rho^0\boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{\nabla} p'\right) - 2\rho^0|\boldsymbol{\nabla} h|\frac{du^0}{dh}\frac{\partial v'_h}{\partial x} = \frac{D^0\dot{\theta}'}{Dt} - \rho^0\boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{f}'\right) \tag{160}$$

Note that this equation still contains the lateral component of the velocity perturbation v'_h, i.e. we have not yet arrived at one equation for the one variable pressure p'. Before taking the next step to eliminate v'_h let us look at the special case where it drops out identically, such that no further manipulation is required.

3.1.1 Uniform flow

For uniform flow $\frac{du^0}{dh} \equiv 0$ in equation (160) and we have

$$\frac{1}{a_0^2}\frac{D_\infty^2 p'}{Dt^2} - \rho^0 \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{\nabla} p'\right) = Q_p \tag{161}$$

$$Q_p = \overbrace{\underbrace{\left(1-\frac{\sigma^0 p_\infty}{\rho^0 T^0}\right)\frac{D_\infty \dot{m}'}{Dt} + \frac{\sigma^0}{T^0}\frac{D_\infty \dot{\vartheta}'}{Dt}}_{\left[\frac{\gamma-1}{(a^2)^0}\frac{D_\infty\dot{\vartheta}'}{Dt}+\frac{1}{\gamma}\frac{D_\infty \dot{m}'}{Dt}\right]_{pg}}}^{=\frac{D_\infty \dot{\theta}'}{Dt}} - \rho^0\boldsymbol{\nabla}\cdot\left(\frac{1}{\rho^0}\boldsymbol{f}'\right)$$

which is called *convected wave equation* (German: "konvektive Wellengleichung"). It describes the propagation of sound in a uniformly moving medium generally including density variations normal to the streamlines. Note that the convected wave equation is almost identical to the wave equation in a medium at rest (60) except that the partial time derivative is replaced by the material time derivative following a streamline of the parallel mean flow: $\frac{\partial}{\partial t} \rightarrow \frac{D_\infty}{Dt} := \frac{\partial}{\partial t} + u_\infty \frac{\partial}{\partial x}$. If the mean flow is not aligned with the $x-$axis, but given as an arbitrary vector $\boldsymbol{v}_\infty$, equation (161) is still valid with $\frac{D_\infty}{Dt} := \frac{\partial}{\partial t} + \boldsymbol{v}_\infty\cdot\boldsymbol{\nabla}$. A consequence of that is, that the convected wave equation may be transformed to the wave equation for a medium at rest simply by introducing a new independent variable $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{u}_\infty t$, whose origin is moving along with the flow.

The same replacement of the time derivatives transfers all equations which we derived for the perturbation dynamics in a medium at rest to those in a medium in uniform motion. Particularly, we may introduce a velocity potential ϕ according to (64) for uniformly moving medium implying that the pressure is related like

$$p' = -\rho_0 \frac{D_\infty \phi}{Dt} + \phi_f \tag{162}$$

Then again, the velocity potential would be governed by equation (68) modified to a convective wave equation by replacing the partial time derivatives by the material time derivatives.

For the special case of uniform mean density $\rho^0 = \rho_\infty$ and therefore $a_0 = a_\infty$ the convected wave equation (161) reduces to

$$\frac{1}{a_\infty^2}\frac{D_\infty^2 p'}{Dt^2} - \Delta p' = Q_p, \tag{163}$$

which upon Fourier transformation (23) yields the convective Helmholtz equation

$$-(k - i\boldsymbol{M}\cdot\boldsymbol{\nabla})^2\, \hat{p} - \Delta\hat{p} = \hat{Q}_p, \tag{164}$$

where $\boldsymbol{M} := \boldsymbol{v}_\infty / a_\infty$ is called "acoustic Mach number", while the hat denotes the Fourier transform (23) of the respective quantity.

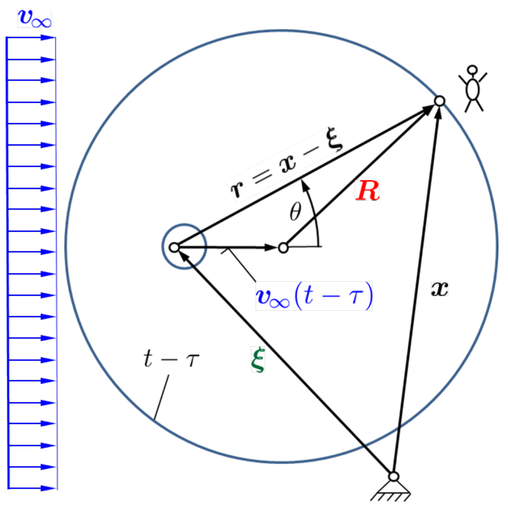

Figure 24: Pulse propagating in a uniform flow field.

3.1.1.1 Green's function for uniformly moving medium For the solution of the convected wave equation for any given source distribution it is very helpful to know the respective Green's function. We restrict ourselves to media with constant $\rho^0 = \rho_\infty$ and $a_0 = a_\infty$. The Green's function satisfies by definition:

$$\frac{1}{a_\infty^2}\frac{D_\infty^2 G_0^f}{Dt^2} - \Delta G_0^f = \delta(\boldsymbol{x}-\boldsymbol{\xi})\delta(t-\tau) \tag{165}$$

We try to reduce the problem of finding G_0^f to the Green's function for a medium at rest. Figure *24* shows a spherical pulse which was sent out at τ at $\boldsymbol{\xi}$. After a period of $t-\tau$ it has spread to a radius R. The propagation of the signal takes place in a medium moving at $\boldsymbol{v}^0 =: \boldsymbol{v}_\infty = u_\infty \boldsymbol{e}_x$. Therefore the origin of the spherical wave has moved by $\boldsymbol{v}_\infty(t-\tau)$. The observer at $\boldsymbol{x}$ cannot distinguish whether the sound pulse arriving at his position came from the (fixed) source point $\boldsymbol{\xi}$ in the moving medium or from a virtual source point at $\boldsymbol{\xi} + \boldsymbol{v}_\infty(t-\tau)$ in a medium at rest. Therefore formally we may immediately write down the Green's function as we know it from a sound propagation problem in a medium at rest. We only have to replace the distance between source and observer r by the distance between the virtual source and the observer R:

$$G_0^f = \frac{\delta(\overbrace{\tau - t + R/a_\infty}^{:=g})}{4\pi R} \tag{166}$$

where now in contrast to the case of a medium at rest the distance

$$R = |\underbrace{\boldsymbol{x}-\boldsymbol{\xi}}_{=\boldsymbol{r}} - \boldsymbol{v}_\infty(t-\tau)| \tag{167}$$

is time dependent. Clearly, only those situations are of interest, where the argument of the Green's function $g = \tau - t + R/a_\infty$ vanishes. The implicit dependence of g on the propagation period $t-\tau$ through R can be expressed explicitly thanks to (114)

$$\delta(g(\tau)) = \sum_{\tau_i(g=0)} \frac{\delta(\tau-\tau_i)}{\left|\frac{dg}{d\tau}\right|_{\tau_i}}, \quad \tau_i \text{ - } i\text{'th zero of } g(\tau)$$

which leads to expressing G_0^f as

$$G_0^f = \sum_{\tau_i(g=0)} \frac{\delta(\tau - t + R(\tau_i)/a_\infty)}{4\pi R(\tau_i)\left|\frac{dg}{d\tau}\right|_{\tau_i}} \tag{168}$$

It is therefore necessary to determine τ_i with

$$g = \tau_i - t + R(\tau_i)/a_\infty = 0 \tag{169}$$

Introducing the acoustic Mach number vector $\boldsymbol{M} = \boldsymbol{v}_\infty/a_\infty$ and its magnitude $M = |\boldsymbol{M}|$ this is equivalent to

$$(t-\tau_i)^2 + \frac{2\boldsymbol{r}\cdot\boldsymbol{v}_\infty}{a_\infty^2(1-M^2)}(t-\tau_i) - \frac{r^2}{a_\infty^2(1-M^2)} = 0$$

Finally we obtain formally two zeros, i.e. two contributions to the sum in (168):

$$\tau_{1/2} = t - \frac{1}{a_\infty}\Big(- \frac{\boldsymbol{r}\cdot\boldsymbol{M}}{(1-M^2)} \pm \frac{\sqrt{(\boldsymbol{r}\cdot\boldsymbol{M})^2 + (1-M^2)r^2}}{(1-M^2)}\Big) \tag{170}$$

which we may equivalently express in terms of the angle θ between $\boldsymbol{e}_r = \boldsymbol{r}/r$ (line of sight between observer and source) and flow direction $\boldsymbol{e}_M = \boldsymbol{M}/M$ (see figure *24*):

$$\tau_{1/2} = t - \frac{1}{a_\infty}\Big(- M\cos\theta \pm \sqrt{1 - M^2\sin^2\theta}\Big)\frac{r}{(1-M^2)} \tag{171}$$

For convenience we now introduce a new distance variable

$$\begin{aligned} r^\pm := R(\tau_{1/2}) &= -\frac{\boldsymbol{r}\cdot\boldsymbol{M}}{(1-M^2)} \pm \frac{\sqrt{(\boldsymbol{r}\cdot\boldsymbol{M})^2 + (1-M^2)r^2}}{(1-M^2)} \\ &= \Big(- M\cos\theta \pm \sqrt{1 - M^2\sin^2\theta}\Big)\frac{r}{(1-M^2)} \end{aligned} \tag{172}$$

Finally we have to determine $\frac{dg}{d\tau}$ in (168). From the definition $g = \tau - t + R/a_\infty$ we have

$$\frac{dg}{d\tau} = 1 + \frac{1}{a_\infty}\underbrace{\frac{\partial R}{\partial \tau}}_{\frac{\boldsymbol{r}\cdot\boldsymbol{v}_\infty - (t-\tau)u_\infty^2}{R}}$$

Evaluated at the zeros $\tau_{1/2}$ as required in (168) we have

$$\begin{aligned} \frac{dg}{d\tau}\Big|_{\tau_{1/2}} &= \frac{R(\tau_{1/2}) + \frac{1}{a_\infty}\Big(\boldsymbol{r}\cdot\boldsymbol{v}_\infty - (t-\tau_{1/2})u_\infty^2\Big)}{R(\tau_{1/2})} = \frac{r^\pm(1-M^2) + \boldsymbol{r}\cdot\boldsymbol{M}}{r^\pm} \\ \Longrightarrow \left|\frac{dg}{d\tau}\right|_{\tau_{1/2}} &= \frac{\sqrt{(\boldsymbol{r}\cdot\boldsymbol{M})^2 + (1-M^2)r^2}}{r^\pm} \end{aligned}$$

If we use this in (168) while remembering that $R(\tau_{1/2}) = r^\pm$ we end up with

$$G_0^f = \frac{\delta(\tau - t + r^+/a_\infty) + \delta(\tau - t + r^-/a_\infty)}{4\pi\sqrt{(\boldsymbol{r}\cdot\boldsymbol{M})^2 + (1-M^2)r^2}} = \frac{\delta(\tau - t + r^+/a_\infty) + \delta(\tau - t + r^-/a_\infty)}{4\pi r\sqrt{(M^2\cos^2\theta + (1-M^2)}} \tag{173}$$

This is the Green's function for the convected wave equation. Note that causality needs to be checked on the Green's function. This means, that the observation of a signal always takes place later than its sending: $t - \tau > 0$. Since $t - \tau = r^{\pm}/a_\infty$ this in turn is equivalent to requiring that $r\pm > 0$. In order to make this requirement more explicit it is convenient to distinguish between subsonic ($M < 1$) and supersonic ($M > 1$) flows.

Subsonic flows.

Since $(1 - M^2) > 0$ the root in (172) is strictly of larger magnitude compared to $\boldsymbol{r}\boldsymbol{\cdot}\boldsymbol{M}$, which on the one hand ensures that $r^+ > 0$, but which on the other hand shows that $r^- < 0$. Therefore, only the r^+- (or τ^+) solution provides a causal component and we have to abandon the r^--part of the solution:

$$G_0^{sub} = \frac{\delta(\tau - t + r^+/a_\infty)}{4\pi\sqrt{(\boldsymbol{r}\boldsymbol{\cdot}\boldsymbol{M})^2 + (1 - M^2)r^2}} = \frac{\delta(\tau - t + r^+/a_\infty)}{4\pi r\sqrt{(M^2\cos^2\theta + (1 - M^2)}} \qquad (174)$$

is the Green's function for the subsonic convected wave equation. Compared to the free field Green's function for a medium at rest ($M = 0$) G_0^{sub} displays a directionally dependent factor $D(\theta)$ in the denominator:

$$D(\theta) = \frac{1}{\sqrt{(M^2\cos^2\theta + (1 - M^2)}} = \frac{1}{\sqrt{1 - M^2\sin^2\theta}} \geq 1 \qquad (175)$$

which (for $M < 1$) is the equation for an ellipsoid. Correspondingly one calls the subsonic convected wave equation of *elliptic type* (German: "elliptischer Typ"). Compared to a medium at rest the presence of the flow generates a directionally dependent amplification of the sound radiation from any source. In particular, there is no amplification in or against the flow direction $D(\theta = 0, \pi) = 1$, while the maximum amplification is observed in the lateral direction $D(\theta = \pm\pi/2) = (1 - M^2)^{-1/2}$, see figure 25.

Knowing the 3D Green's function $G_0(x - \xi, y - \eta, z - \zeta, t - \tau)$ for the convective wave equation, it is now quite easy to obtain the corresponding 2D Green's function. Upon integrating the

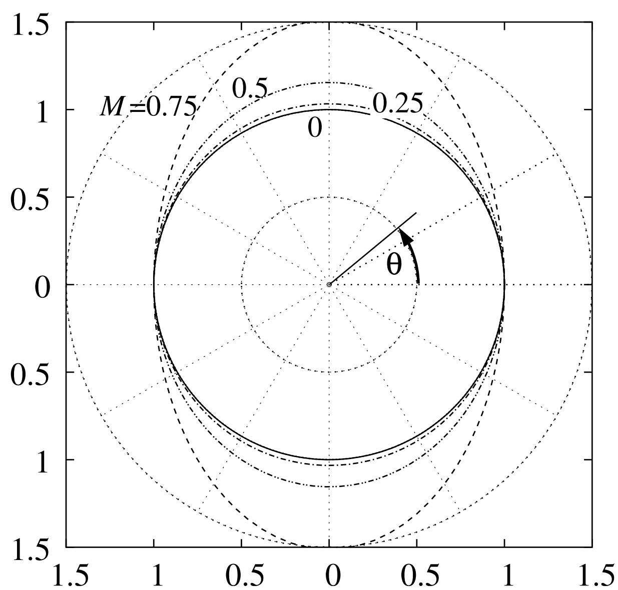

Figure 25: Directivity $D(\theta)$ of subsonic Green's function G_0^{sub} for various Mach numbers.

convective wave equation (165) along a direction, say z, perpendicular to the flow direction, i.e. $\int_{-\infty}^{\infty} \ldots d\zeta$ we convert the 3D problem to a 2D problem. Correspondingly $G_0^{(2D)} = \int_{-\infty}^{\infty} G_0 d\zeta$, which results in (see appendix B for details)

$$G_0^{(2D)} = \frac{H\left(t - \tau + \frac{r}{a_\infty}\frac{M_r - \sqrt{1 - M^2 + M_r^2}}{1 - M^2}\right)}{2\pi\sqrt{1-M^2}\sqrt{\left[t - \tau + \frac{r}{a_\infty}\frac{M_r}{1-M^2}\right]^2 - \frac{r^2}{a_\infty^2}\frac{(1 - M^2 + M_r^2)}{(1-M^2)^2}}} \qquad (B.7)$$

where $M_r := \boldsymbol{M} \cdot \boldsymbol{r}/r$ while $\boldsymbol{r} = (x-\xi)\boldsymbol{e}_x + (y-\eta)\boldsymbol{e}_y$ with $r = |\boldsymbol{r}|$ and H denotes the Heaviside function.

The 1D Green's function is similarly derived from the 3D Green's function by integration over the plane $x = 0$ perpendicular to the flow direction, i.e. $G_0^{(1D)} = \int_0^{2\pi}\int_0^{\infty} G_0 \; R\, dR\, d\varphi$ with $R^2 = (y-\eta)^2 + (z-\zeta)^2$ and φ the circumferential direction around the x-axis (details see appendix B):

$$G_0^{(1D)} = \frac{a_\infty}{2} H\left(t - \tau - \frac{r}{a_\infty}\frac{1}{1+M_r}\right) \qquad (B.9)$$

where $\boldsymbol{r} = (x - \xi)\boldsymbol{e}_x$.

Supersonic flows.

Since $(1 - M^2) < 0$ the root in (172) is strictly of smaller magnitude compared to the one of $\boldsymbol{r}\cdot\boldsymbol{M}$, such that $r^{\pm} > 0$, but only downstream of the source (i.e. $\boldsymbol{r}\cdot\boldsymbol{M} > 0$). This in turn means, the contributions of r^+ as well as r^- are both not causal upstream but causal downstream. But even then, the value of $r^{\pm}$ needs to be a real number. Let us check, under what conditions this is the case. From (172) we find that a real solution is possible only for

$$1 - M^2 \sin^2\theta > 0 \Longleftrightarrow \sin\theta < \frac{1}{M} \text{ or } \sin\theta > -\frac{1}{M}$$

This condition is valid for both contributions to the supersonic Green's function G_0^{super}. Having sorted out any upstream signals already above, there exists no solution (complete silence) exterior

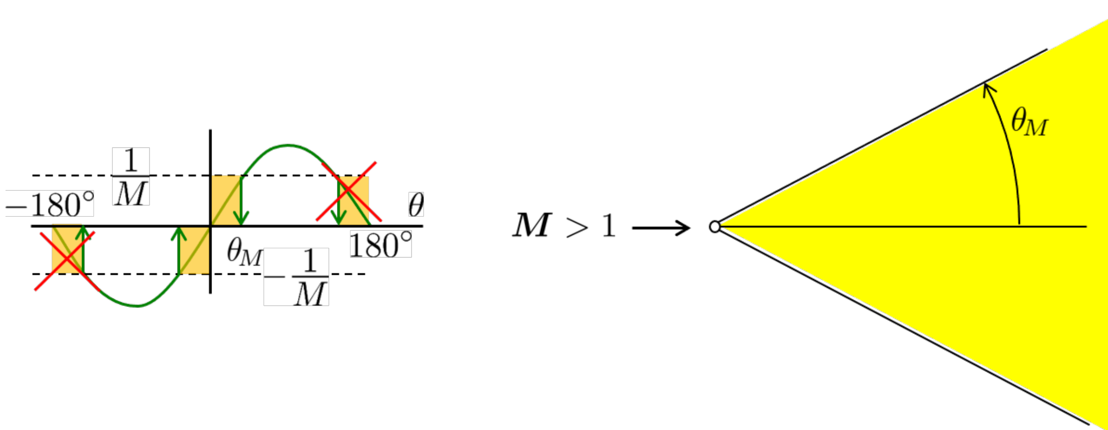

Figure 26: Mach cone, no solution outside cone, 2 solutions inside cone.

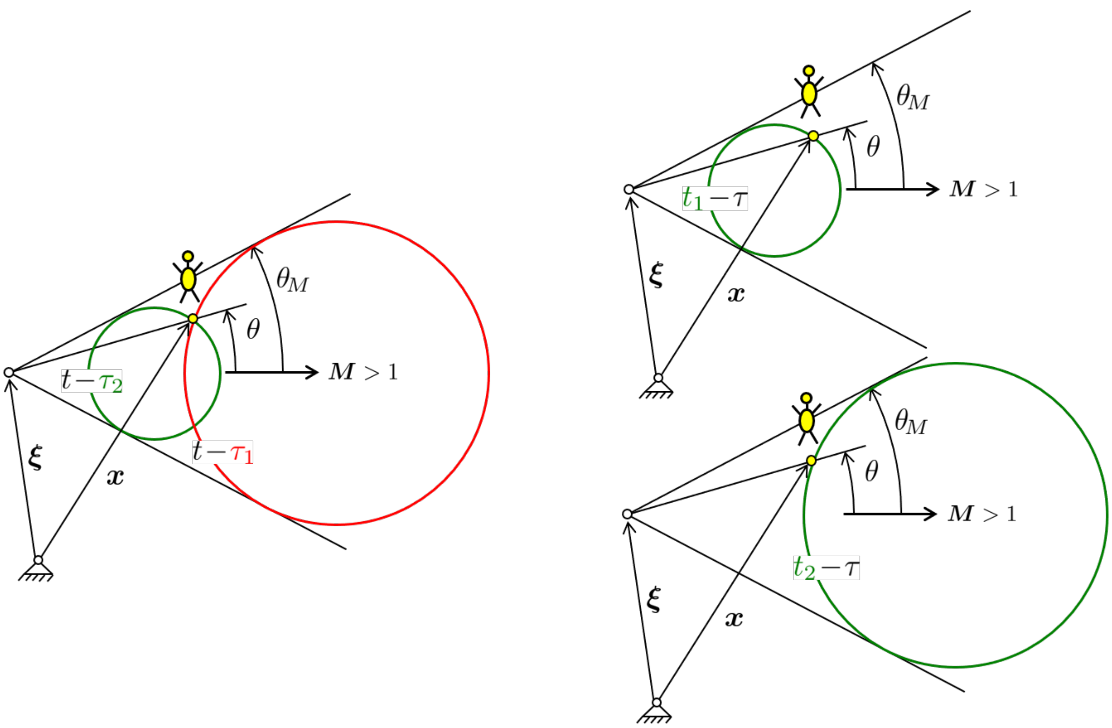

Figure 27: Two signals from one pulse at supersonic flow. Left: two signals from different source times received simultaneous at receiver position, Right: two signals received at different reception times from one source pulse.

to the cone defined by $-\theta_M < \theta < \theta_M$ with $\theta_M = \arcsin(M^{-1})$; inside this cone there are two separate contributions to the solution. This cone is called *Mach cone* (German "Machscher Kegel") and the angle θ_M is called *Mach angle* (German: "Machscher Winkel"), see figure *26*. Again, inside the cone there exist two contributions at any reception point. Either two pulses fired at different points in time reach the reception point simultaneaously (see left of figure *27*), or a pulse fired at one point in time results in two pulses received at different times at the reception point (see right of figure *27*). Note, that the left part of figure*27* corresponds to the case of a point source traveling through quiescent air from right to left with a the same Mach number. The two circles would correspond to pulses emitted when the source passed through the respective centers of the circles. This means that a stationary observer would first receive the boom at arrival time of the Mach cone followed by a signal simultaneously composed of sound radiated by the source in forward and rearward direction.

For the Green's function we have (173). As for the subsonic Greens function we observe a distinct directivity $D(\theta)$ given in (175), which represents hyperbolae for $M > 1$ (see figure *28*). Correspondingly the supersonic convected wave equation is called to be of *hyperbolic type* (German "hyperbolischer Typ").

3.1.1.2 Sound field of point sources

Sound field of a point mass or heat source in subsonic flow

The convected wave equation for the sound pressure (161) shows the acoustic source terms of

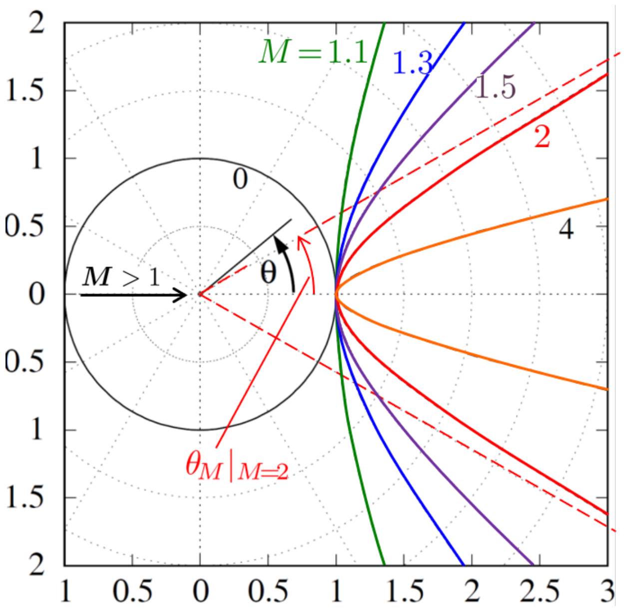

Figure 28: Directivity $D(\theta)$ of supersonic Green's function G_0^{super} for various Mach numbers.

mass- and heat sources $\dot{\theta}'$ to take the form $Q_p = \dfrac{D_\infty \dot{\theta}'}{Dt}$. We assume uniform speed of sound $a_0 = a_\infty$ (and uniform density $\rho^0 = \rho_\infty$) of the medium. Our non-moving source $\dot{\theta}'$ is assumed to be concentrated at point $\boldsymbol{\xi}_0$. We may therefore describe it as $\dot{\theta}'(\boldsymbol{\xi}, \tau) = \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)\theta_p(\tau)$, where $\theta_p(\tau)$ denotes the temporal dependence of the source. The sound field as solution to the convected wave equation is written down immediately according to the Green's function method: Multiplication of the source by the Green's function of the problem and integration over space and time:

$$p'(\boldsymbol{x}, t) = \int\limits_{-\infty}^{\infty} \int\limits_{V_\infty} G_0^f \left(\frac{\partial}{\partial \tau} + u_\infty \frac{\partial}{\partial \xi} \right) [\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)\theta_p(\tau)] \, dV(\boldsymbol{\xi}) d\tau$$

We split the integral into the part p_a containing the local time derivate and p_b containing the convective derivative and substitute the expression for the Green's function to obtain

$$p'(\boldsymbol{x}, t) = \underbrace{\int\limits_{-\infty}^{\infty} \frac{\partial \theta_p}{\partial \tau} \frac{\delta(\tau - t + r_0^+/a_\infty)}{4\pi r_0 \sqrt{1 - M^2 \sin^2 \theta_0}} d\tau}_{=:\, p_a} + \underbrace{\int\limits_{V_\infty} u_\infty \frac{\partial}{\partial \xi} [\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)] \frac{\theta_p(t - r^+/a_\infty)}{4\pi r \sqrt{1 - M^2 \sin^2 \theta}} dV(\boldsymbol{\xi})}_{=:\, p_b}$$

with $\sin^2\theta = 1 - (\boldsymbol{r} \cdot \boldsymbol{M})^2/(rM)^2$ and $r = |\boldsymbol{x} - \boldsymbol{\xi}|$ and r^+ according to (172). The volume integral in p_a and the time integral in p_b are trivial and have been carried out already. The quanties with subscript zero are meant to be evaluated at $\boldsymbol{\xi} = \boldsymbol{\xi}_0$, e.g. $r_0 = |\boldsymbol{x} - \boldsymbol{\xi}_0|$.

The first integral is solved by writing

$$p_a = \frac{1}{4\pi r_0 \sqrt{1 - M^2 \sin^2 \theta_0}} \left[\int\limits_{-\infty}^{\infty} \frac{\partial}{\partial \tau} \left(\theta_p \delta(\tau - t + r_0^+/a_\infty) \right) d\tau - \int\limits_{-\infty}^{\infty} \theta_p(\tau) \underbrace{\frac{\partial \delta}{\partial \tau}}_{-\frac{\partial \delta}{\partial t}} d\tau \right]$$

The first of these two time integrals can be carried out explicitly and yields zero. In the second integral advantage is taken of the fact that the delta function depends on τ in the same way as on t. Therefore it is possible to replace $\frac{\partial\delta}{\partial\tau}$ by $-\frac{\partial\delta}{\partial t}$ and drag the time derivative before the integral, which gives

$$p_a = \frac{1}{4\pi r_0\sqrt{1-M^2\sin^2\theta_0}}\frac{\partial}{\partial t}\int\limits_{-\infty}^{\infty}\theta_p(\tau)\delta(\tau-t+r_0^+/a_\infty)d\tau = \frac{1}{4\pi r_0\sqrt{1-M^2\sin^2\theta_0}}\frac{\partial\theta_p}{\partial t}\Big|_{t-r_0^+/a_\infty}$$

For the solution for p_b we first replace $u_\infty\frac{\partial}{\partial\xi}$ equivalently by $\boldsymbol{v}_\infty\cdot\boldsymbol{\nabla}_\xi$ and express the integral by use of the product rule in the integrand like

$$\begin{aligned} p_b &= \frac{1}{4\pi}\int\limits_{V_\infty}\boldsymbol{\nabla}_\xi\cdot\left[\boldsymbol{v}_\infty\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\frac{\theta_p(t-r^+/a_\infty)}{\sqrt{(1-M^2)r^2+M^2(x-\xi)^2}}\right]dV(\xi) \\ &- \frac{1}{4\pi}\int\limits_{V_\infty}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\underbrace{\boldsymbol{\nabla}_\xi\cdot\left[\boldsymbol{v}_\infty\frac{\theta_p(t-r^+/a_\infty)}{\sqrt{(1-M^2)r^2+M^2(x-\xi)^2}}\right]}_{-\boldsymbol{\nabla}_x\cdot[\ldots]}dV(\xi) \end{aligned}$$

Again, the first integral vanishes, while in the second integral the $\boldsymbol{\nabla}_x$ operation may be dragged before the integral. Furthermore $\boldsymbol{v}_\infty$ -being a constant- may be taken out of the integral and we may again replace $\boldsymbol{v}_\infty\cdot\boldsymbol{\nabla}_x = u_\infty\frac{\partial}{\partial x}$, leaving:

$$p_b = \frac{u_\infty}{4\pi}\frac{\partial}{\partial x}\int\limits_{V_\infty}\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\frac{\theta_p(t-r^+/a_\infty)}{\sqrt{(1-M^2)r^2+M^2(x-\xi)^2}}dV(\boldsymbol{\xi}) = \frac{u_\infty}{4\pi}\frac{\partial}{\partial x}\left[\frac{\theta_p(t-r_0^+/a_\infty)}{\sqrt{(1-M^2)r_0^2+M^2(x-\xi_0)^2}}\right]$$

Next, the actual differentiation with respect to x has to be executed, i.e.

$$p_b = \frac{u_\infty}{4\pi\sqrt{(1-M^2)r_0^2+M^2(x-\xi_0)^2}}\underbrace{\frac{\partial\theta_p}{\partial x}}_{\frac{\partial\theta_p}{\partial t}\left(-\frac{1}{a_\infty}\right)\frac{\partial r_0^+}{\partial x}} - \frac{u_\infty\theta_p}{4\pi}\frac{x-\xi_0}{\sqrt{(1-M^2)r_0^2+M^2(x-\xi_0)^2}^3}$$

which gives:

$$\begin{aligned} p_b &= -\frac{M}{4\pi r_0(1-M^2)\sqrt{1-M^2\sin^2\theta_0}}\left[\frac{\cos\theta_0}{\sqrt{1-M^2\sin^2\theta_0}}-M\right]\frac{\partial\theta_p}{\partial t}\Big|_{t-r_0^+/a_\infty} \\ &- \frac{a_\infty M\cos\theta_0}{4\pi r_0^2\sqrt{1-M^2\sin^2\theta_0}^3}\,\theta_p\big|_{t-r_0^+/a_\infty} \end{aligned}$$

Finally $p' = p_a + p_b$ yields

$$\begin{aligned} p'(\boldsymbol{x},t) &= -\frac{a_\infty M\cos\theta_0}{4\pi r_0^2\sqrt{1-M^2\sin^2\theta_0}^3}\,\theta_p\big|_{t-r_0^+/a_\infty} + \\ &+ \frac{1}{4\pi r_0}\underbrace{\frac{1}{(1-M^2)\sqrt{1-M^2\sin^2\theta_0}}\left[1-\frac{M\cos\theta_0}{\sqrt{1-M^2\sin^2\theta_0}}\right]}_{=:\,D}\frac{\partial\theta_p}{\partial t}\Big|_{t-r_0^+/a_\infty} \end{aligned} \quad (176)$$

If this result is compared to the sound field of a mass- and heat source in a non-moving medium, i.e. $p'(M=0) = \dfrac{1}{4\pi r_0}\dfrac{\partial \theta_p}{\partial t}$, one observes that
(a) the flow generates a directed (θ-dependent) near field, represented by the first term in (176), which decays fast like r_0^{-2}
(b) the flow generates a *directivity* (German "Richtwirkung") D on the farfield (the term decaying slowly like r_0^{-1}). The directivity D is plotted in the left diagram of fig.*29* for various flow Mach numbers.
For an observer on a streamline through the source point $\boldsymbol{\xi}_0$, who is located downstream of the source, i.e. $\theta_0 = 0$, the amplitude of the sound signal is reduced by $D(\theta_0 = 0) = (1+M)^{-1}$. An observer located on that streamline but upstream ($\theta_0 = \pi$) senses a signal whose strength is increased by $D(\theta_0 = \pi) = (1-M)^{-1}$. This flow effect is therefore commonly called *convective amplification* (German "konvektive Verstärkung").

We could have arrived at our pressure field (176) as well by means of a velocity potential according to (162). The corresponding velocity potential of the point source in uniform flow of constant density $\rho^0 = \rho_\infty$ is

$$\phi(\boldsymbol{x}, t) = -\frac{\theta_p|_{t - r_0^+/a_\infty}}{4\pi\rho_\infty r_0\sqrt{1 - M^2\sin^2\theta_0}} \tag{177}$$

The advantage of knowing the potential is of course that now we have direct access to the acoustic particle velocity $\boldsymbol{v}' = \boldsymbol{\nabla}\phi$. The velocity of the point source in constant mean flow according to (177) is given by (*C.1*) in appendix C for reference.

The acoustic field for a point mass or heat source in subsonic flow in 1D and 2D is listed in the appendix C in equations (*C.3*) and (*C.2*) respectively.

Contour plots of the instantaneous sound pressure distribution of a sinusoidal mass- or heat source

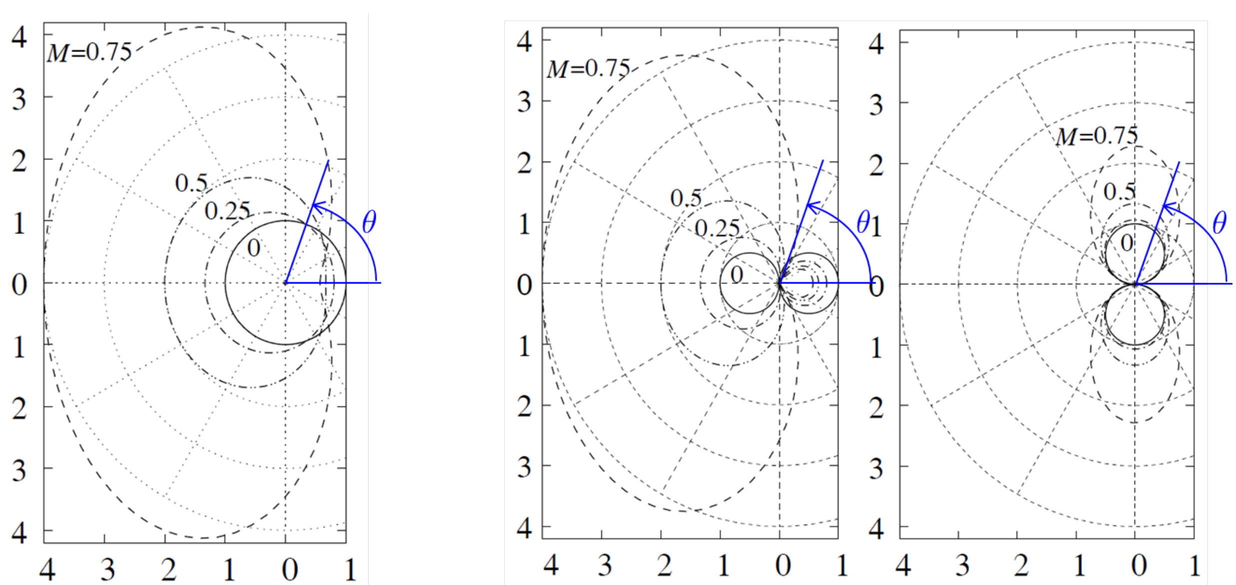

Figure 29: Left: directivity D of a mass- or heat point source in uniform mean flow, centre: directivity $|D_x|$ and right: directivity $|D_h|$ of point force in uniform flow.

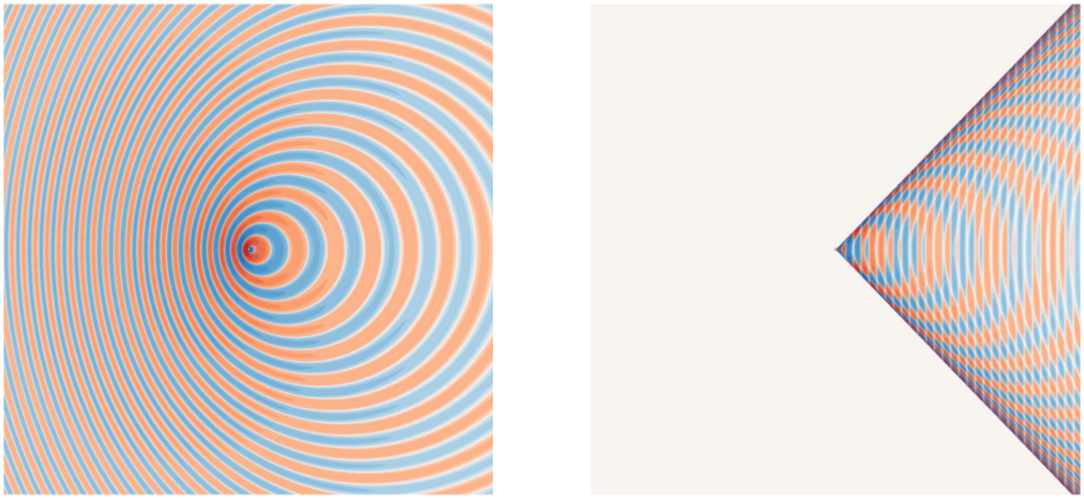

Figure 30: Sound pressure field of a sinusoidal mass- or heat point source in uniform mean flow, left: $M = 0.5$, right: $M = 1.5$. Quasi-logarithmic presentation by consecutive doubling of colormap increments.

in a flow field is depicted in figure *30*. The left part of the figure shows a subsonic flow ($M = 0.5$) from left to right, the right part shows a supersonic case ($M = 1.5$). The subsonic situation clearly shows the contraction of the wavelengths upstream and a respective stretching downstream. Moreover, the intensity of the contour colors appears increased upstream compared to downstream, indicating the convective amplication effects discussed before. The most striking observation in the supersonic situation (figure *30* right) is the fact that a sound field exists only inside of the Mach cone. Here we see a complicated interference pattern of superimposed waves since there are two waves sent out at different times impacting simultaneously at each given reception point.

Sound field of a point force in subsonic flow

The convected wave equation for the sound pressure (161) shows the acoustic source term due to external forces $\boldsymbol{f}'$ to take the form $Q_p = -\boldsymbol{\nabla}\cdot\boldsymbol{f}'$. We assume uniform speed of sound $a_0 = a_\infty$ (and uniform density $\rho^0 = \rho_\infty$) of the medium. Our non-moving force $\boldsymbol{f}'$ is assumed to be concentrated at point $\boldsymbol{\xi}_0$. We may therefore describe it as $\boldsymbol{f}'(\boldsymbol{\xi},\tau) = \delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\boldsymbol{f}_p(\tau)$, where $\boldsymbol{f}_p(\tau)$ denotes the temporal dependence of the source. The sound field as solution to the convected wave equation is written down immediately according to the Green's function method: Multiplication of the source by the Green's function of the problem and integration over space and time:

$$p'(\boldsymbol{x},t) = -\int_{-\infty}^{\infty}\int_{V_\infty} G_0^f \boldsymbol{\nabla}_\xi\cdot \left[\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\boldsymbol{f}_p(\tau)\right] dV(\boldsymbol{\xi})d\tau$$

Analogously as in the previous example we may re-write this integral applying the product rule to the integrand:

$$p'(\boldsymbol{x},t) = -\int_{-\infty}^{\infty}\int_{V_\infty} \boldsymbol{\nabla}_\xi\cdot \left[G_0^f\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\boldsymbol{f}_p(\tau)\right] - \delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0)\boldsymbol{f}_p(\tau)\cdot\boldsymbol{\nabla}_\xi G_0^f dV(\boldsymbol{\xi})d\tau$$

Applying Gauss' theorem one can see that the first integral vanishes. The evaluation of the second integral is straight forward and we finally arrive at

$$\begin{aligned} p'(\boldsymbol{x},t) &= \frac{1}{4\pi r_0^2\sqrt{1-M^2\sin^2\theta_0}^3}\left[M^2\cos\theta_0\boldsymbol{e}_M+(1-M^2)\boldsymbol{e}_r\right]\cdot\boldsymbol{f}_p\big|_{t-r_0^+/a_\infty}+ \\ &+\frac{1}{4\pi a_\infty r_0}\frac{1}{(1-M^2)\sqrt{1-M^2\sin^2\theta_0}}\left[\frac{M^2\cos\theta_0\boldsymbol{e}_M+(1-M^2)\boldsymbol{e}_r}{\sqrt{1-M^2\sin^2\theta_0}}-M\boldsymbol{e}_M\right]\cdot\frac{\partial\boldsymbol{f}_p}{\partial t}\Big|_{t-r_0^+/a_\infty} \end{aligned} \quad (178)$$

As for the mass- or heat source we recognize the first term to be the near field part of the pressure perturbation, which decays fast with r_0^{-2}, while the second term represents the farfield part, decaying slowly like r_0^{-1}.

We are mostly interested in the farfield part of the solution. In order to still interpret the solution (178) better, we represent the force vector $\boldsymbol{f}_p$ by its streamwise component f_{pM} and its lateral component f_{ph}, i.e.

$$\boldsymbol{f}_p = f_{pM}\boldsymbol{e}_M + f_{ph}\boldsymbol{e}_h$$

and insert it into the farfield part of (178). Respecting $\boldsymbol{e}_M\cdot\boldsymbol{e}_r = \cos\theta_0$ and $\boldsymbol{e}_M\cdot\boldsymbol{e}_h = 0$ and $\boldsymbol{e}_h\cdot\boldsymbol{e}_r = \sin\theta_0$ we obtain for the farfield due to the point force

$$\begin{aligned} p'(\boldsymbol{x},t) &= \frac{1}{4\pi a_\infty r_0}\underbrace{\frac{1}{(1-M^2)\sqrt{1-M^2\sin^2\theta_0}}\left[\frac{\cos\theta_0}{\sqrt{1-M^2\sin^2\theta_0}}-M\right]}_{=:D_M}\frac{\partial f_{pM}}{\partial t}\Big|_{t-r_0^+/a_\infty}+ \\ &+\frac{1}{4\pi a_\infty r_0}\underbrace{\frac{\sin\theta_0}{1-M^2\sin^2\theta_0}}_{=:D_h}\frac{\partial f_{ph}}{\partial t}\Big|_{t-r_0^+/a_\infty} \end{aligned}$$

We recognize different directivities depending on whether the force is acting in the streamwise direction or the lateral direction. The directivities D_M and D_h are plotted in the two right diagrams of fig.*29* for various flow Mach numbers.

3.1.1.3 Wave propagation through duct flows We consider the sound propagation of a sound wave along an infinitely long duct of constant rectangular cross section of height d_z and width d_y (see figure *31*). The medium of constant density $\rho^0 = \rho_\infty$ and speed of sound $a^0 = a_\infty$ in the duct is moving uniformly with $\boldsymbol{v}^0 = \boldsymbol{v}_\infty = u_\infty\boldsymbol{e}_x$ along the duct axis $\boldsymbol{e}_x$. The duct surfaces are assumed acoustically hard, implying that the wall-normal component of the particle velocity must vanish: $\boldsymbol{n}\cdot\boldsymbol{v}' = 0$, where $\boldsymbol{n}$ denotes the normal wall vector. For technical flows of this type it may be quite important to describe the behavior of sound waves traveling in the duct, especially the way in which sound is transmitted through the duct.

The sound propagation is described by the homogeneous form ($Q_p = 0$) of the convected wave equation (161), in this case with constant speed of sound a_0 and constant density ρ^0:

$$\frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t}+u_\infty\frac{\partial}{\partial x}\right)^2 p' - \Delta p' = 0$$

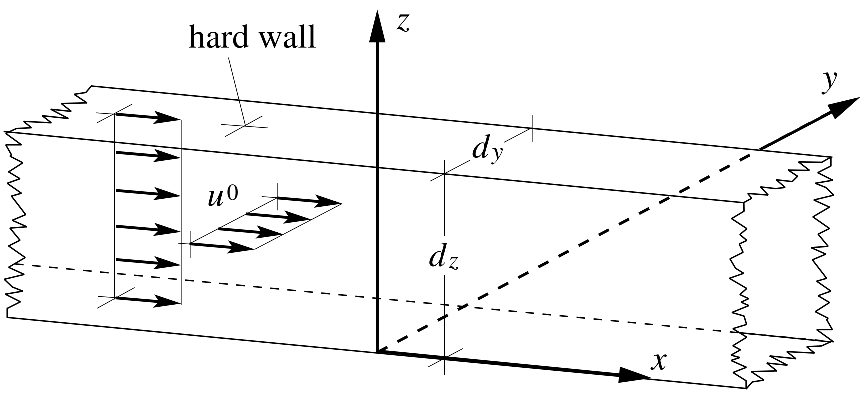

Figure 31: Definition of a duct geometry.

The boundary condition of vanishing normal wall component of the velocity perturbation may be translated into an equivalent condition on the pressure perturbation when multiplying (159) by the unit normal wall vector $\boldsymbol{n}$. This leaves the same condition as in the medium at rest, namely $\frac{\partial p'}{\partial n} = 0$ at the walls.

The problem is homogeneous in the direction x and t, i.e. the boundary conditions are not a function of x or t. Therefore we may separate out the solution dependence on x and t (separation ansatz) and we may further assume an exponential form in t:

$$p'(t,x,y,z) = p^t(t)p^x(x)p^{yz}(y,z), \qquad \text{with } p^t(t) = \exp(i\omega t) \tag{179}$$

for a given frequency ω. Upon substitution into the convected wave equation and re-arrangement such that purely x-dependent terms appear on the left hand side while the y,z-dependent terms appear on the right hand side we have

$$\frac{1}{p^x}\left[\left(i\frac{\omega}{a_\infty} + M\frac{d}{dx}\right)^2 - \frac{d^2}{dx^2}\right]p^x = \frac{1}{p^{yz}}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)p^{yz} =: -C^2 \tag{180}$$

Where we have introduced the separation constant C, which is to be determined later when satisfying the boundary conditions. From (180) we may extract two separate equations for p^x and p^{yz}, which are coupled to one another through C:

$$\left[\left(\frac{\omega}{a_\infty}\right)^2 - C^2\right]p^x - 2i\frac{\omega}{a_\infty}M\frac{dp^x}{dx} + (1-M^2)\frac{d^2p^x}{dx^2} = 0 \tag{181}$$

$$\frac{\partial^2 p^{yz}}{\partial y^2} + \frac{\partial^2 p^{yz}}{\partial z^2} + C^2 p^{yz} = 0 \tag{182}$$

This last may again be solved using a separation between the y-and z-dependence:

$$p^{yz}(y,z) = p^y(y)p^z(z) \tag{183}$$

by substitution into (182) and subsequent re-arrangement we have

$$\underbrace{\frac{1}{p^y}\frac{d^2p^y}{dy^2}}_{=:\,-C_y^2} + \underbrace{\frac{1}{p^z}\frac{d^2p^z}{dz^2}}_{=:\,-C_z^2} = -C^2 \tag{184}$$

The reason for introducing the new constants C_y and C_z is due to the fact, that the left hand side of equation (184) can only be equal to a constant if the re-grouped y-dependent terms themselves as well as the re-grouped z-dependent terms themselves form constants. The equations for p^y and p^z are trivial to solve and give:

$$\begin{aligned} p^y &= A^y \sin(C_y y) + B^y \cos(C_y y) \\ p^z &= A^z \sin(C_z z) + B^z \cos(C_z z) \end{aligned} \tag{185}$$

The various constants are determined such that the boundary condition of a hard wall

$$\frac{dp^y}{dy}(y = 0, d_y) = 0, \quad \frac{dp^z}{dz}(z = 0, d_z) = 0 \tag{186}$$

are satisfied. For the y-direction:

$$\begin{aligned} \frac{dp^y}{dy}(y = 0) &= C_y A^y = 0 && \Longrightarrow A^y = 0 \\ \frac{dp^y}{dy}(y = d_y) &= -C_y B^y \sin(C_y d_y) = 0 && \Longrightarrow C_y = m\pi/d_y \quad m = 1, 2, 3, \ldots \end{aligned}$$

This shows, that there are infinitely many solutions for p^y which we denote $p^y_m = B^y_m \cos(ym\pi/d_y)$. Analogously we obtain

$$A^z = 0, \quad C_z = n\pi/d_z \quad n = 1, 2, 3, \ldots$$

or $p^z_n = B^z_n \cos(zn\pi/d_z)$ respectively. Having determined $C_y(m)$ and $C_z(n)$ the separation constant $C^2 = C_y^2 + C_z^2$ is

$$C^2_{nm} = \pi^2(m^2/d_y^2 + n^2/d_z^2) \tag{187}$$

Combining the two constants B^y_m and B^z_n to a new constant $P_{mn} = B^y_m B^z_n$ the function p^{yz} reads:

$$p^{yz}_{mn} = P_{mn} \cos\Big(m\frac{\pi}{d_y} y\Big) \cos\Big(n\frac{\pi}{d_z} z\Big) \tag{188}$$

The function characterized by the selection of a specific m and n is called *mode* (German: "Mode"). We now come to determine p^x from (181) to describe the x-dependence of the solution. We have to solve an ordinary linear differential equation in x with constant coefficients which is done by using an exponential ansatz $p^x = \exp(-i\alpha x)$, which turns (181) into

$$\Big[\Big(\frac{\omega}{a_\infty}\Big)^2 - \pi^2\Big(\frac{m^2}{d_y^2} + \frac{n^2}{d_z^2}\Big)\Big] - 2\frac{\omega}{a_\infty} M\alpha - (1 - M^2)\alpha^2 = 0 \tag{189}$$

being the dispersion relation of the problem which establishes the relation between the wavenumber x-component α and the frequency ω. Solving for α we get

$$\alpha^\pm_{mn} = \frac{1}{1 - M^2}\Big\{ -\frac{\omega}{a_\infty} M \pm \sqrt{\frac{\omega^2}{a_\infty^2} - \pi^2\Big(\frac{m^2}{d_y^2} + \frac{n^2}{d_z^2}\Big)(1 - M^2)}\Big\} \tag{190}$$

where the indices m, n denote the dependence of α on the choice of the mode numbers. Eqn (190) shows that for each mode number there exist two solutions. Finally we may put together the overall pressure field to be

$$p' = \sum_{m,n=0}^{\infty} P_{mn}^{\pm} \exp(i\omega t - i\alpha_{mn}^{\pm} x) \cos\left(m\frac{\pi}{d_y} y\right) \cos\left(n\frac{\pi}{d_z} z\right) \tag{191}$$

Behavior for $M = 0$:

In this case (190) gives

$$\alpha_{mn}^{\pm} = \pm\sqrt{\frac{\omega^2}{a_\infty^2} - \pi^2\left(\frac{m^2}{d_y^2} + \frac{n^2}{d_z^2}\right)} \tag{192}$$

For a chosen mode m, n and according to (191) we thus obtain for large enough fixed ω a right running $(+)$ and a left running mode wave $(-)$ with the same wavelength. It is important to notice that for a given frequency wave propagation ceases to exist when the mode numbers exceed a certain critical value. This happens when the term under the root becomes negative, i.e.

$$f_c^{mn} = \omega_c/2\pi < \frac{a_\infty}{2}\sqrt{\frac{m^2}{d_y^2} + \frac{n^2}{d_z^2}} \tag{193}$$

f_c^{mn} is called *cut-off* frequency of mode m, n. A mode changes its character from propagational to non-propagational when the considered frequency is below the cut-off frequency. In order to qualify this change to exponential behavior it may be shown, that the pressure along the duct axis x is strictly decaying away from the location where the signal was generated. From (193) it may be seen that the $m, n = 0, 0$ mode (plane wave), is always propagational. The lowest of the higher order modes (either $m, n = 1, 0$ or $m, n = 0, 1$) becomes cut-off for a frequency

$$f_{crit} = \omega_{crit}/2\pi = \frac{a_\infty}{2\min(d_y, d_z)} \tag{194}$$

below which none of the infinitely many higher order modes may be transmitted through the duct. This critical frequency is a property of the duct geometry. Figure *32* shows the result of a numerical simulation of the sound signals generated by a diaphragm in a turbulent 2D duct flow. The diaphragm acts as source of broadband noise. The microphones upstream receive a signal, strongly affected by the cut-on behavior of the duct. Below 1700Hz only plane waves are radiated from the source. The pressure levels show a significant increase beyond the critical cut-on frequencies. This increase occurs because more ”acoustic degrees of freedom” are excited by the source. Note, that due to the shape of mode $n = 1$ the centre microphone does not see the contributions of this mode to the spectrum.

The same cut-off phenomena as in rectangular ducts are observed in circular ducts and the cut-off criterion is used extensively in the design of aeroengines. The rotating fan produces no plane waves in the duct of the engine’s nacelle. Therefore one can choose e.g. the fan blade and stator blade numbers such that the frequencies produced in the turbo engine relative to the dimensions of the nacelle are cut-off. In this way a very effective noise reduction is achieved. We come to discuss these issues in detail in the subsequent lecture.

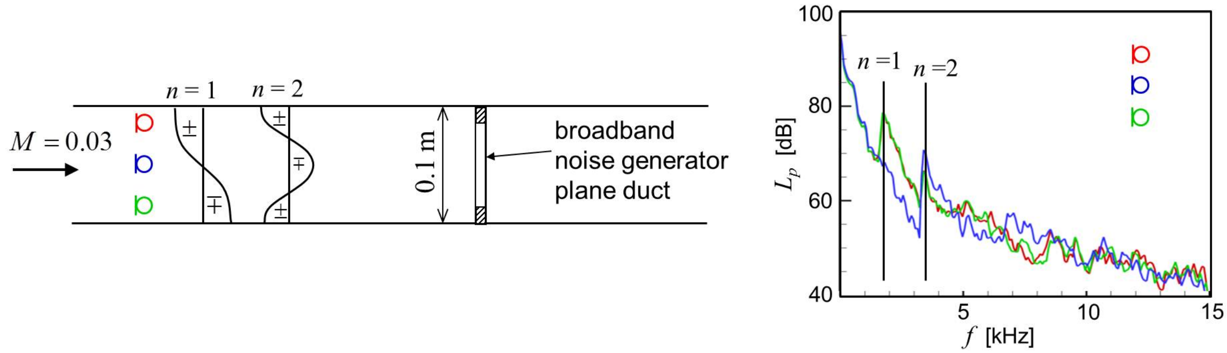

Figure 32: Propagation of a broadband signal, generated by a diaphragm in a plane (2D) duct with turbulent flow ($M \approx 0$). Influence of cut-on effects on spectral shape, cut-on frequencies according to (193) with $m = 0$, $a_\infty = 340m/s$.

Behavior for $M \neq 0$:

According to (190) the cut-off frequency is now

$$f_c^{mn} = \omega_c/2\pi < \frac{a_\infty}{2}\sqrt{\frac{m^2}{d_y^2} + \frac{n^2}{d_z^2}}\sqrt{1 - M^2} \qquad (195)$$

For subsonic flows $M < 1$ this clearly shows that the influence of the mean flow through the duct is to reduce the cut-off frequency when compared to the no-flow case. This means that for given frequency more modes are transmitted through the duct.

The plane wave mode ($0, 0$-mode) shows most easily the effect of the flow on the wavenumber. We have

$$\alpha_{00}^{+} = \frac{1}{1+M}\frac{\omega}{a_\infty} \quad \text{and } \alpha_{00}^{-} = -\frac{1}{1-M}\frac{\omega}{a_\infty} \qquad (196)$$

This means that α^+ represents a wave running along the flow direction while α^- characterizes a wave running against the flow direction. The wavelength $\lambda = 2\pi/\alpha$ of the upstream running wave is compressed by a factor $1 - M$, while the wavelength of the downstream running wave is stretched by a factor $1 + M$.

3.1.2 Kirchhoff integral for uniform flow

Having the instrument of the free field Green's function for uniform flow available, we are now able to generalize the Kirchhoff integral (141) for the case that the acoustic medium is in uniform motion. As for the no-flow case the Kirchhoff integral determines the sound pressure field outside a closed surface ∂V_H (see figure 33), including all source domains V_s and objects V_B, assuming the field is known on the surface. The derivation is analogous to the one in 2.6.3.

Again, we define the domain V_H inside of ∂V_H with the help of the scalar indicator function $f(\boldsymbol{x})$ in (140) whose zero-level surface is identical with ∂V_H, and with the normalization that $|\boldsymbol{\nabla} f|_{f=0} = 1$. As before we introduce a new pressure variable $\underline{p}' = H(f)p'$ now applying the

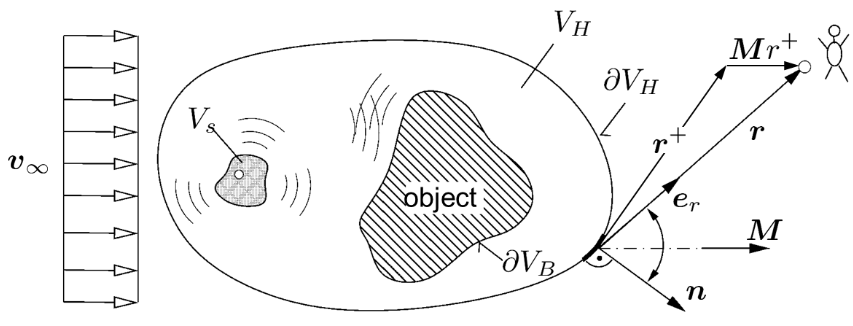

Figure 33: *Fixed Kirchhoff integration surface ∂V_H in uniform flow.*

convected wave operator $\frac{1}{a_\infty^2}\frac{D_\infty^2}{Dt^2} - \Delta$ on the right and left hand sides of this definition equation. After elementary application of the product and chain rules of differentiation we obtain

$$\frac{1}{a_\infty^2}\frac{D_\infty^2 \underline{p}'}{Dt^2} - \Delta \underline{p}' = \overbrace{HQ_p}^{=0} - \boldsymbol{\nabla}\cdot(p'\delta(f)\boldsymbol{\nabla} f) - \delta(f)\boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} p' + $$
$$+\frac{1}{a_\infty^2}\Big[2\delta(f)\boldsymbol{v}_\infty\cdot\boldsymbol{\nabla} f\frac{\partial p'}{\partial t} + \boldsymbol{v}_\infty\boldsymbol{v}_\infty\boldsymbol{:}\Big(\boldsymbol{\nabla}(p'\delta(f)\boldsymbol{\nabla} f) + \delta(f)\boldsymbol{\nabla} f\boldsymbol{\nabla} p'\Big)\Big]$$

in which the second line of the right hand side is new when compared to the respective no-flow equation. We solve the above convective wave equation by the Green's function method, i.e. multiply by the 3D free field convective Green's function G_0^f according to (174) and integrate over all space V_∞ and all time:

$$\underline{p}' = \int_{-\infty}^{\infty}\int_{V_\infty} -[\boldsymbol{\nabla}_\xi\cdot(p'\delta(f)\boldsymbol{n}) + \delta(f)\boldsymbol{n}\cdot\boldsymbol{\nabla}_\xi p']G_0^f +$$
$$+\frac{1}{a_\infty^2}\Big[2\delta(f)\boldsymbol{v}_\infty\cdot\boldsymbol{n}\frac{\partial p'}{\partial\tau} + \boldsymbol{v}_\infty\boldsymbol{v}_\infty\boldsymbol{:}(\boldsymbol{\nabla}_\xi(p'\delta(f)\boldsymbol{n}) + \delta(f)\boldsymbol{n}\boldsymbol{\nabla}_\xi p')\Big]G_0^f \;\; dV(\xi)\,d\tau,$$

where we have used the fact that $\boldsymbol{\nabla}_\xi f = \boldsymbol{n}$ is the unit normal vector pointing outside of ∂V_H. Next we form divergence expressions in order to obtain $\delta(f)$ outside of the derivatives for the solution of the volume integral:

$$\underline{p}' = \int_{-\infty}^{\infty}\int_{V_\infty} \frac{1}{a_\infty}\Big[2\delta(f)\boldsymbol{M}\cdot\boldsymbol{\nabla}_\xi f\Big(\frac{\partial G_0^f p'}{\partial\tau} - p'\frac{\partial G_0^f}{\partial\tau}\Big) +$$
$$+\boldsymbol{\nabla}_\xi\cdot\Big[\boldsymbol{M}\boldsymbol{M}\delta(f)p'G_0^f\boldsymbol{n}\Big] - \delta(f)p'\boldsymbol{M}\cdot\boldsymbol{n}\boldsymbol{M}\cdot\boldsymbol{\nabla}_\xi G_0^f + \delta(f)G_0^f\boldsymbol{M}\boldsymbol{M}\boldsymbol{:}(\boldsymbol{n}\boldsymbol{\nabla}_\xi p') -$$
$$-\boldsymbol{\nabla}_\xi\cdot[\delta(f)p'G_0^f\boldsymbol{n}] + \delta(f)p'\boldsymbol{n}\cdot\boldsymbol{\nabla}_\xi G_0^f - \delta(f)G_0^f\boldsymbol{n}\cdot\boldsymbol{\nabla}_\xi p' \;\; dV(\xi)\,d\tau.$$

Since $G_0^f = G_0^f(t-\tau)$ we replace its τ-derivative by the negative t-derivative in the first line of the right hand side. The two leading divergence expressions in the second and third line of the right hand side integrate to zero by the Gauss-theorem and the assumption that one will not let the Kirchhoff surface extend to infinity. Exchanging time and volume integration for the first term in the first line yields again zero. The volume integral may then be taken for all terms to give

$$\underline{p'} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{\partial V_H} \frac{2}{a_\infty}\boldsymbol{M\cdot n}\, p' G_0^f \, dS(\xi)\, d\tau + \\ + \int_{-\infty}^{\infty}\int_{\partial V_H} p'(-\boldsymbol{M\cdot n}\ \boldsymbol{M\cdot} + \boldsymbol{n\cdot})\boldsymbol{\nabla}_\xi G_0^f + G_0^f(\boldsymbol{MM{:}}(\boldsymbol{n\nabla}_\xi p') - \boldsymbol{n\cdot\nabla}_\xi p')\, dS(\xi)\, d\tau.$$

Now the actual expression of the convective Green's function G_0^f from (174), i.e.

$$G_0^f = \frac{\delta(\tau - t + r^+/a_\infty)}{4\pi r^*}, \quad r^* = \sqrt{(\boldsymbol{M\cdot r})^2 + (1-M^2)\boldsymbol{r}^2}, \quad r^+ = \frac{-\boldsymbol{r\cdot M} + r^*}{1-M^2}$$

along with it's spatial derivative

$$\boldsymbol{\nabla}_\xi G_0^f = \frac{1}{4\pi a_\infty r^*(1-M^2)}\left\{\boldsymbol{M} - \frac{(\boldsymbol{r\cdot M})\boldsymbol{M} + (1-M^2)\boldsymbol{r}}{r^*}\right\}\frac{\partial\delta}{\partial\tau} + \frac{(\boldsymbol{r\cdot M})\boldsymbol{M} + (1-M^2)\boldsymbol{r}}{4\pi r^{*3}}\,\delta$$

are substituted into the solution to finally arrive at the *convected Kirchhoff integral*

$$\underline{p'} = \frac{1}{4\pi}\int_{\partial V_H}\frac{1}{a_\infty r^*}\left(M_n + \frac{\boldsymbol{r\cdot n}}{r^*}\right)\frac{\partial p'}{\partial t} + \frac{(1-M^2)\boldsymbol{r\cdot n}}{r^{*3}}p' + \frac{M_n}{r^*}\boldsymbol{M\cdot\nabla}_\xi p' - \frac{1}{r^*}\frac{\partial p'}{\partial n}\, dS(\xi), \quad (197)$$

where $M_n := \boldsymbol{n\cdot M}$ denotes the surface normal component of the (assumed subsonic) acoustic Mach number of the uniformly moving medium. The Kirchhoff integral is written for a closed integration surface at rest. The observer at $\boldsymbol{x}$ is at rest as well. The normal vector $\boldsymbol{n}$ on the surface element $dS(\xi)$ is by definition pointing towards the exterior of the surface. The integrals need to be evaluated at the retarded time $\tau = t - r^+/a_\infty$. Note that the only convection effect linear in the Mach number M enters through the time derivative of the pressure on the Kirchhoff surface; all other convection effects are quadratic in M, which become important only at high subsonic flow speeds. Note also that in contrast to the no-flow Kirchhoff integral, now not only the normal derivative of the pressure on the surface is needed but also the derivative along the flow $\boldsymbol{M\cdot\nabla}_\xi p'$.

3.1.3 Parallel shear flows

In our derivation of an equation for the pressure perturbation for a general parallel mean flow with shear we arrived at relation (160). For flows with non-zero shear this equation still contains the lateral component of the velocity perturbation v_h', which still needs to be eliminated. Although an expression for v_h' is not available, we may find one for its material time derivative. We multiply the momentum equation (159) by the lateral unit vector $\boldsymbol{e}_h = \boldsymbol{\nabla}h/|\boldsymbol{\nabla}h|$, which is independent of t and x:

$$\frac{D^0 v_h'}{Dt} = \frac{1}{\rho^0}\Big(-\underbrace{\boldsymbol{e}_h\boldsymbol{\cdot\nabla}p'}_{\frac{\partial p'}{\partial h}} + \underbrace{\boldsymbol{e}_h\boldsymbol{\cdot f'}}_{f_h'}\Big) \quad (198)$$

Now we take the material time derivative D^0/Dt of (160) yielding

$$\frac{D^0}{Dt}\Big[\frac{1}{a_0^2}\frac{D^{02}p'}{Dt^2} - \rho^0\boldsymbol{\nabla}\cdot\Big(\frac{1}{\rho^0}\boldsymbol{\nabla}p'\Big)\Big] - 2\rho^0|\boldsymbol{\nabla}h|\frac{du^0}{dh}\frac{\partial}{\partial x}\Big(\frac{D^0v_h'}{Dt}\Big) = \frac{D^{02}\dot{\theta}'}{Dt^2} - \rho^0\frac{D^0}{Dt}\Big[\boldsymbol{\nabla}\cdot\Big(\frac{1}{\rho^0}\boldsymbol{f}'\Big)\Big]$$

Now we substitute our above expression for D^0v_h'/Dt and obtain the pressure perturbation equation for parallel shear flows

$$\frac{D^0}{Dt}\Big[\frac{1}{a_0^2}\frac{D^{02}p'}{Dt^2} - \rho^0\boldsymbol{\nabla}\cdot\Big(\frac{1}{\rho^0}\boldsymbol{\nabla}p'\Big)\Big] + 2|\boldsymbol{\nabla}h|\frac{du^0}{dh}\frac{\partial^2 p'}{\partial h\partial x} = Q_{ps} \tag{199}$$

$$Q_{ps} = \frac{D^{02}\dot{\theta}'}{Dt^2} + 2|\boldsymbol{\nabla}h|\frac{du^0}{dh}\frac{\partial f_h'}{\partial x} - \rho^0\frac{D^0}{Dt}\Big[\boldsymbol{\nabla}\cdot\Big(\frac{1}{\rho^0}\boldsymbol{f}'\Big)\Big]$$

There are three characteristics to mention about this equation:

1. Contrary to the 2^{nd} order wave equations which we derived previously (199) is of 3^{rd} order. The consequence is that this equation not only contains acoustic degrees of freedom, but additionally a vortical degree of freedom. Although we cannot express this perturbation vortex dynamics explicitly this indicates a form of pressure which is not linked to the compressibility like the acoustic pressure and which would exist for a perfectly incompressible fluid as well. Moreover for many shear flows (especially those with a turning point in their velocity profile) the non-acoustic pressure perturbations tend to get amplified exponentially with time. Such flows are called hydrodynamically unstable. This means that the vortical degrees of freedom may represent instabilities. Even without excitation $Q_{ps} = 0$ there may exist non-trivial solutions for p' which do not decay in time. In fact, (199) is nothing but a special form of the so called *Rayleigh equation* known in hydrodynamic stability analysis. In that context it is used to identify inviscid hydrodynamic instabilities.

2. The shear of the mean flow profile plays a role in the sound generation (see term $2|\boldsymbol{\nabla}h|\frac{du^0}{dh}\frac{\partial f_h'}{\partial x}$)

3. For uniform flow (or even zero flow) equation (199) does not reduce to the convected (or simple) wave equation (161) or (60) respectively.

3.1.3.1 Propagation of a plane sound wave through a plane parallel shear layer We consider a plane, free shear layer as in the left part of figure *34*. The shear (and temperature) layer's extension in the lateral direction z is $-\delta < z < \delta$. A plane harmonic sound of wavelength $\lambda_{-\infty}$ approaches the shearlayer from far below ($z \approx -\infty$) under an angle of incidence ϑ_i (see sketch on the right of figure *34*). An observer at a position $z \ll -\delta$ far from the shear layer detects a change of phase (frequency $f = \omega/(2\pi)$). Only the component of the wave motion normal to the phase line (front) contributes to the generation of a phase change:

$$(a_{-\infty}^0 + u_{-\infty}^0\cos\vartheta_i) = \lambda_{-\infty}f \tag{200}$$

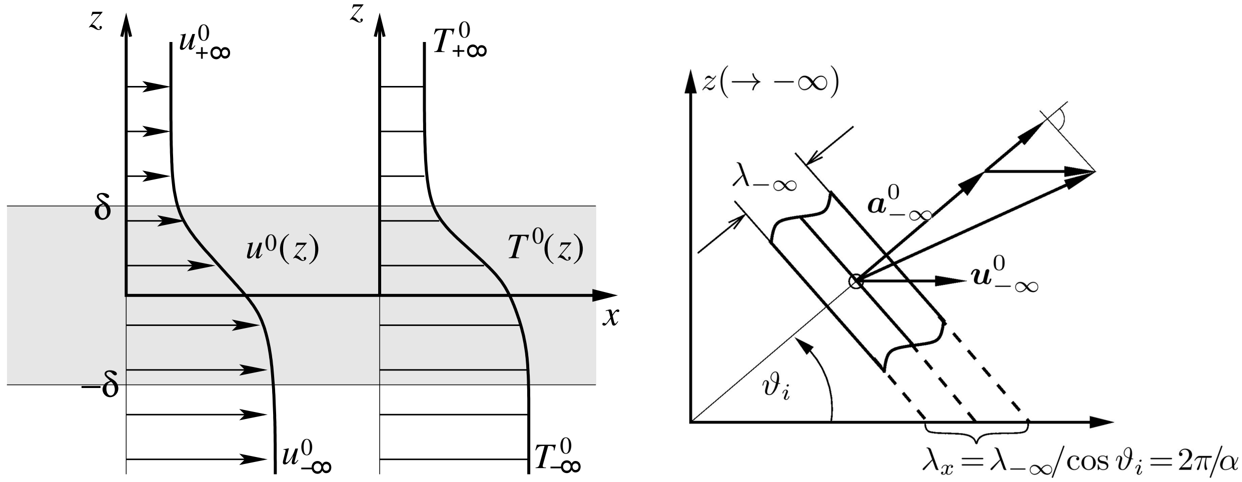

Figure 34: Left: definition of a parallel shear- and temperature layer, right: plane harmonic sound wave incident from below the shearlayer.

We express this relation in terms of the wavenumber $k_{-\infty} = 2\pi/\lambda_{-\infty}$ and ω:

$$\frac{\omega}{a^0_{-\infty} k_{-\infty}} = 1 + M_{-\infty}\cos\vartheta_i, \quad M_{-\infty} := \frac{u^0_{-\infty}}{a^0_{-\infty}} \tag{201}$$

In order to describe the propagation of the sound wave through the shearlayer we employ equation (199). Since we have a plane shear layer $h(y, z) = z$ and therefore $\boldsymbol{\nabla} h = \boldsymbol{e}_z$ and $|\boldsymbol{\nabla} h| = 1$. We are only interested in the propagation of the sound wave, i.e. $Q_{ps} = 0$. Then we have from (199)

$$\frac{D^0}{Dt}\Big[\frac{1}{a_0^2}\frac{D^{02}p'}{Dt^2} - \underbrace{\frac{\gamma p^0}{a_0^2}}_{\rho^0}\boldsymbol{\nabla}\cdot\Big(\underbrace{\frac{a_0^2}{\gamma p^0}}_{(\rho^0)^{-1}}\boldsymbol{\nabla} p'\Big)\Big] + 2\frac{du^0}{dz}\frac{\partial^2 p'}{\partial z \partial x} = 0$$

where we have substituted ρ^0 using the equation of state of a perfect gas. Note that γp^0 is constant and cancels. Finally we obtain

$$\frac{1}{a_0^2}\frac{D^0}{Dt}\Big[\frac{D^{02}p'}{Dt^2} - \boldsymbol{\nabla}\cdot\Big(a_0^2\boldsymbol{\nabla} p'\Big)\Big] + 2\frac{du^0}{dz}\frac{\partial^2 p'}{\partial z \partial x} = 0 \tag{202}$$

Since the problem's boundary and initial conditions do not depend on t and x we may again solve the equation with an exponential ansatz

$$p' = \hat{p}(z)\exp(i\omega t - i\alpha x), \quad \alpha = k_{-\infty}\cos\vartheta_i \tag{203}$$

where α and ω are given numbers. Substitution into (202) yields:

$$\frac{d^2\hat{p}}{dz^2} + 2\Big(\frac{1}{a_0}\frac{da_0}{dz} + \frac{\alpha}{\omega - \alpha u^0}\frac{du^0}{dz}\Big)\frac{d\hat{p}}{dz} + \Big(\Big[\frac{\omega}{a_0} - \alpha M^0\Big]^2 - \alpha^2\Big)\hat{p} = 0 \tag{204}$$

This is a second order linear differential equation with non-constant coefficients and needs to be solved numerically. Note, that (204) very much resembles the equation of motion of a damped

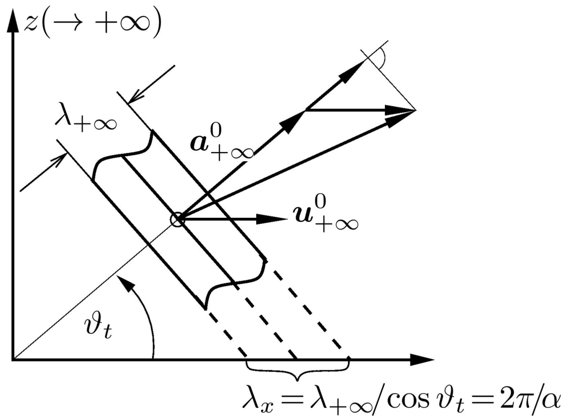

Figure 35: Description of a plane wave after transmission through the shearlayer

harmonic oscillator when we identify z with time t. We know that for such an oscillator the factor before the first derivative is to be understood as damping coefficient. This indicates that the shear and the temperature gradient (here expressed as da_0/dz) tend to decrease or increase the wave's amplitude $\hat{p}$ depending on the sign of their gradient.

Let us look at the asymptotic forms of the equations for $z \longrightarrow \pm\infty$ and the respective solutions.

For $z \ll -\delta$ we have from (204) which is easily solved to yield

$$\frac{d^2\hat{p}^-}{dz^2} + \underbrace{\left(\left[\frac{\omega}{a^0_{-\infty}} - \alpha M_{-\infty}\right]^2 - \alpha^2\right)}_{(k_{-\infty}\sin\vartheta_i)^2}\hat{p}^- = 0 \tag{205}$$

$$\hat{p}^- = p^i \exp(-i|k_{-\infty}\sin\vartheta_i|z) + p^r \exp(i|k_{-\infty}\sin\vartheta_i|z) \tag{206}$$

Obviously we get two solutions. Their physical behavior can be seen from (203). The common time factor is $\exp(i\omega t)$, i.e. (for assumed $\omega > 0$) the first term in (206) represents a wave traveling towards increasing z (towards the shearlayer). This is obviously the incident wave which we consider given. The second term on the other hand represents a wave traveling from the layer towards $z = -\infty$. This shows that the equation allows for wave components which are not transmitted through the shearlayer, but get reflected.

Further note that for supersonic flows $M_{-\infty} > 1$ the frequency ω could not always be chosen as some positive number. In this case there exists some critical incidence angle $\vartheta_i^c := \arccos(-1/M_{-\infty})$ beyond which ω would have to be negative according to (201). This critical angle corresponds again to the Mach cone angle plus 90^o. What wave propagation would a negative ω describe according to (203) and (206)? The incident part going with p^i in (206) would now describe a wave propagating not in the direction $\vartheta_i > \vartheta_i^c$, but in the respective anti-direction, i.e. coming from the shearlayer. This shows that there cannot exist any incident sound wave for incidence angles beyond ϑ_i^c.

In $z > \delta$ we expect to find the transmitted sound wave. For $z \gg \delta$ we have from (204)

$$\frac{d^2\hat{p}^+}{dz^2} + \left(\left[\frac{\omega}{a^0_{+\infty}} - \alpha M_{+\infty}\right]^2 - \alpha^2\right)\hat{p}^+ = 0 \tag{207}$$

Equation (207) is easily solved and has the form

$$\hat{p}^{+} = p^{t}\exp(-i\beta z) + p^{\infty}\exp(i\beta z) \tag{208}$$

where β due to (207),may be expressed explicitly substituting ω from (201) and α from (203):

$$\beta = k_{-\infty}\sqrt{\left[\frac{a^0_{-\infty}}{a^0_{+\infty}}(1 + M_{-\infty}\cos\vartheta_i) - M_{+\infty}\cos\vartheta_i\right]^2 - \cos^2\vartheta_i} \tag{209}$$

Whether the solution (208) really represents a wave in $z > 0$ depends on β. In order to discuss the type of solution more explicitly, we re-write (209) as

$$\beta \;=\; k_{-\infty}\{H(\sigma) - i(1 - H(\sigma))\}\sqrt{|\sigma|} = -ik_{-\infty}\sqrt{-\sigma} \tag{210}$$

$$\sigma = \left[\frac{a^0_{-\infty}}{a^0_{+\infty}}(1 + M_{-\infty}\cos\vartheta_i) - M_{+\infty}\cos\vartheta_i\right]^2 - \cos^2\vartheta_i \tag{211}$$

with H the Heaviside function. The character of the solution in $z > 0$ changes from an oscillatory type for real β (i.e. in combination with the assumed time factor in (203) a wave-like solution) to a monotonous type for imaginary β. Note that for given Mach numbers and temperatures of the mean flow the behavior of the pressure signal depends on the angle under which the sound wave hits the layer ϑ_i. We will come back to this dependence in detail later.

imaginary β:

What, if the mean flow and angle of incidence to the shear layer ϑ_i in (210) are such that $\sigma < 0$? Then the component $p^t\exp(-k_{-\infty}\sqrt{\sigma}z)$ in (208) represents an amplitude decreasing exponentially with distance z from the shearlayer. The second component p^{∞} of the solution in $z > 0$ goes with a function exponentially increasing with the distance from the shearlayer. A pressure signal with an infinitely large amplitude at infinity would be unphysical, i.e. $p^{\infty} = 0$. In summary, for $\sigma < 0$ an incident sound wave is not transmitted through the shear layer at all; the pressure signal dies out for $z > 0$. There occurs a *perfect reflexion* (German: "Totalreflexion") of the incident sound wave at the layer.

real β:

Let us now assume β to be real, i.e. $\sigma > 0$ and $\beta = k_{-\infty}\sqrt{|\sigma|}$. Clearly in this case a sound wave exists in $z > 0$. There can be only the transmitted wave p^t in (208) which travels towards increasing z. The solution component p^{∞} describes a sound wave incident from $z = +\infty$ which we exclude since we are only interested in the transmission problem, i.e. as before $p^{\infty} = 0$.

Having made sure that a wave-like solution is found in $z > 0$ we may describe the solution for large z according to the nomenclature shown in figure *35*. We relate the solutions on both sides of the shearlayer to one another using two quantities, which are conserved across the layer. Since the medium is flowing steadily, the only unsteadiness is given by ω:

$$\omega = \underbrace{(1 + M_{-\infty}\cos\vartheta_i)a^0_{-\infty}k_{-\infty} = (1 + M_{+\infty}\cos\vartheta_t)a^0_{+\infty}k_{+\infty}}$$

Further the wavenumber x-component α is by definition constant for all z:

$$\alpha = \underbrace{k_{-\infty}\cos\vartheta_i = k_{+\infty}\cos\vartheta_t}$$

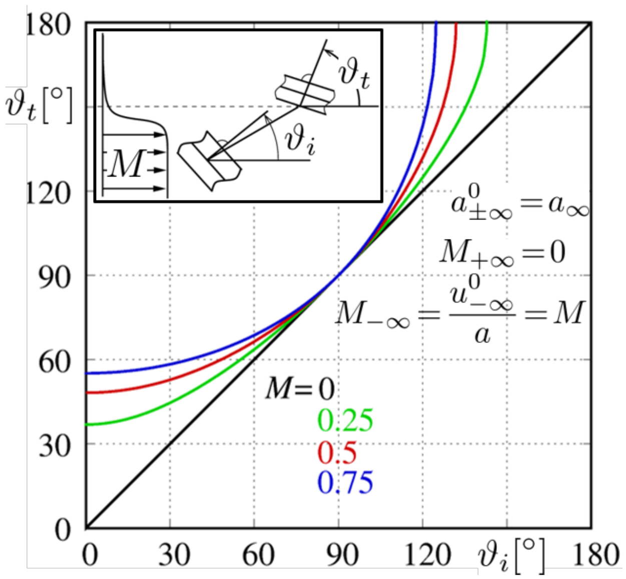

Figure 36: Relation between orientations of incident and transmitted sound wave front for different flow Mach numbers M.

From these two relations we find

$$k_{+\infty} = \frac{(u^0_{-\infty} - u^0_{+\infty})\cos\vartheta_i + a^0_{-\infty}}{a^0_{+\infty}}\, k_{-\infty} \tag{212}$$

$$\cos\vartheta_t = \frac{a^0_{+\infty}}{(u^0_{-\infty} - u^0_{+\infty})\cos\vartheta_i + a^0_{-\infty}}\, \cos\vartheta_i \tag{213}$$

We observe that the wavelength $\lambda = 2\pi/k$ changes at the shearlayer. The change of the radiation direction $\vartheta_i \longrightarrow \vartheta_t$ as result of the temperature- or shearlayer is called *refraction* (German: ”Brechung” or ”Refraktion”). Figure *36* shows the relation (213) for $a^0_{-\infty} = a^0_{+\infty}$ and $u^0_{+\infty} = 0$.

Perfect or specular reflection. (German: ”Totalreflexion”) If the sound wave is bent at the shearlayer in such a way that the transmission direction comes to lie parallel to the flow direction $\vartheta_t \to 0°$ or $\vartheta_t \to 180°$, then no actual transmission takes place. The incident sound is reflected perfectly. The transmission angle $\vartheta_t = 0°$ denotes the limiting case of a wave travelling in the flow direction; the corresponding incidence angle ϑ_i follows from relation (213):

$$\cos\vartheta_i^{tot}(\vartheta_t = 0°) = \frac{a^0_{-\infty}}{u^0_{+\infty} - u^0_{-\infty} + a^0_{+\infty}} \tag{214}$$

Incident waves with an orientation $0 < \vartheta_i < \vartheta_i^{tot}$ are reflected and not transmitted through the layer. For constant temperature, $a^0_{+\infty} = a^0_{-\infty} =: a_0$ we have $\cos\vartheta_i^{tot} = (M_{+\infty} - M_{-\infty} + 1)^{-1}$. Since the magnitude of the cosine may not exceed 1, we find that for an incident wave with a downstream directed propagation orientation the condition $M_{+\infty} > M_{-\infty}$ must be satisfied in order that perfect reflection would occur.

As mentioned above, the other limiting case of perfect reflection is indicated for $\vartheta_t = 180^o$, i.e.

$$\cos\vartheta_i^{tot}(\vartheta_t = 180°) = \frac{a^0_{-\infty}}{u^0_{+\infty} - u^0_{-\infty} - a^0_{+\infty}} \tag{215}$$

Incident waves with an orientation $\vartheta_i^{tot} < \vartheta_i < 180°$ are as well reflected and not transmitted through the layer. For constant temperature, $a^0_{+\infty} = a^0_{-\infty} =: a_0$ we have $\cos\vartheta_i^{tot} = (M_{+\infty} - M_{-\infty} - 1)^{-1}$. Since the magnitude of the cosine may not exceed 1, we find that for an incident wave with an upstream directed propagation orientation the condition $M_{+\infty} < M_{-\infty}$ must be satisfied in order that perfect reflection would occur. Such a case is seen in figure *36*. The diagram shows that for incidence angles larger than $\sim 145^o$ the $M_{-\infty}$=0.25 shearlayer transmits no sound waves; there exist no corresponding ϑ_t.

For constant flow $u^0_{+\infty} = u^0_{-\infty}$ relations (214) and (215) give $\cos\vartheta_i^{tot} = \pm a_{-\infty}/a_{+\infty}$. This tells us, that $a_{-\infty} < a_{+\infty}$ is the condition for which perfect reflection takes place. This is true, when the temperature distribution is such that the incident wave is traveling towards the higher temperature domain.

Zone of silence. (German: "Schallschatten") As a limiting case, the smallest angle under which an incident wave may hit the shearlayer is $\vartheta_i = 0°$; the largest possible angle of incidence is $\vartheta_i = 180°$. The corresponding transmission angles ϑ_t^s determine sectors into which the shearlayer does not transmit any sound. These angle sectors are called *zone of silence*. We determine the angle of the zone of silence of an incident wave running in the flow direction by requiring that $\vartheta_i = 0°$. Then relation (213) gives

$$\cos\vartheta_t^s(\vartheta_i = 0^o) = \frac{a^0_{+\infty}}{u^0_{-\infty} - u^0_{+\infty} + a^0_{-\infty}} \tag{216}$$

The angle sector $0 < \vartheta_t < \vartheta_t^s$ is free of sound, because any incidence angle $> 0°$ would render a $\vartheta_t > \vartheta_t^s$; the refraction mechanism does not allow for sound to enter into the zone of silence. For constant temperature, $a^0_{+\infty} = a^0_{-\infty} =: a_0$ we have $\cos\vartheta_t^s = (M_{-\infty} - M_{+\infty} + 1)^{-1}$. Since the magnitude of the cosine may not exceed 1, we find that for an incident wave with a downstream directed propagation orientation the condition $M_{+\infty} < M_{-\infty}$ must be satisfied in order that a zone of silence would occur. Such a case is seen in figure *36*. The diagram shows that for incidence angle $\vartheta_i = 0°$ the transmission angle is $\vartheta_t^s \sim 35°$ for the $M_{-\infty}$=0.25 shearlayer. This is the smallest possible transmission angle in the diagram and thus determines the zone of silence. The zone of silence grows with the Mach number of the flow.

As mentioned above, the other limiting case of a zone of silence is indicated for an incidence angle $\vartheta_i = 180°$, i.e.

$$\cos\vartheta_t^s(\vartheta_i = 180°) = \frac{a^0_{+\infty}}{u^0_{-\infty} - u^0_{+\infty} - a^0_{-\infty}} \tag{217}$$

The angle sector $\vartheta_t^s < \vartheta_t < 180°$ is free of sound, because any incidence angle $< 180^o$ would render a $\vartheta_t < \vartheta_t^s$. For constant temperature, $a^0_{+\infty} = a^0_{-\infty} =: a_0$ we have $\cos\vartheta_t^s = (M_{-\infty} - M_{+\infty} - 1)^{-1}$. Since the magnitude of the cosine may not exceed 1, we find that for an incident wave with an upstream directed propagation orientation the condition $M_{+\infty} > M_{-\infty}$ must be satisfied in order that a zone of silence would occur.

For constant flow $u^0_{+\infty} = u^0_{-\infty}$ relations (216) and (217) give $\cos\vartheta_t^s = \pm a_{+\infty}/a_{-\infty}$. This tells us, that $a_{-\infty} > a_{+\infty}$ is the condition for which a zone of silence occurs. This is true, when the temperature distribution is such that the incident wave is traveling towards the lower temperature domain.

Change in amplitude. Let us estimate the change in amplitude the wave experiences when it travels across the shear layer, i.e. we have to relate p^t to p^i. We make the simplifying assumption that the shearlayer thickness becomes infinitely small: $\delta \longrightarrow 0$. Then we do have to match $\hat{p}_-(z=-\epsilon)$ from (206) to $\hat{p}_+(z=+\epsilon)$ from (208) for $0<\epsilon\to 0$. At the interface the pressure must be continous; it cannot jump because there is no external force acting:

$$\hat{p}^-(z=0)=\hat{p}^+(z=0) \Longleftrightarrow p^i+p^r=p^t \tag{218}$$

Further it must be guaranteed that the medium adjacent from above and from below would neither intrude into itself nor form voids across the interface, represented by the layer. To arrive at a respective condition at the interface we first describe its position. We may denote the vertical deflection of our (plane) shearlayer by $h(x,t)$. Therefore the location of the layer may be described as $f(x,z,t) := z-h(t,x)\equiv 0$. The substantial derivative is then $\dfrac{Df}{Dt}=-\dfrac{\partial h}{\partial t}+\boldsymbol{v}\cdot(\boldsymbol{e}_z-\boldsymbol{\nabla} h)\equiv 0$. From this it follows that $v_z=\dfrac{\partial h}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla} h$. We assume small deflections $h=h^0+\epsilon h'(t,x)$ with $h^0=0$ the unperturbed shearlayer at $z=0$. Linearization gives $v_z'=\dfrac{\partial h'}{\partial t}+u^0\dfrac{\partial h'}{\partial x}=:\dfrac{D^0h'}{Dt}$. The deflection h' has to be the same for the medium just below or just above the shearlayer; thus $v_z'^+=\dfrac{\partial h'}{\partial t}+u^0_{+\infty}\dfrac{\partial h'}{\partial x}$ or $v_z'^-=\dfrac{\partial h'}{\partial t}+u^0_{-\infty}\dfrac{\partial h'}{\partial x}$. Eliminating h' finally yield the following condition at $z=0$:

$$\Big(\frac{\partial}{\partial t}+u^0_{+\infty}\frac{\partial}{\partial x}\Big)v_z'^-=\Big(\frac{\partial}{\partial t}+u^0_{-\infty}\frac{\partial}{\partial x}\Big)v_z'^+$$

Writing $v_z'^\pm=\hat{v}_z^\pm\exp(i\omega t-i\alpha x)$ the momentum equation (198) helps us to express $\hat{v}_z^\pm$ in terms of the pressure:

$$\hat{v}_z^\pm=\frac{1}{\rho^0_{\pm\infty}}\frac{i}{(\omega-\alpha a^0_{\pm\infty}M_{\pm\infty})}\frac{\partial\hat{p}^\pm}{\partial z}$$

Upon using (201) at $z=0$ the above two relations give

$$\frac{1}{\rho^0_{-\infty}}\frac{ik_{-\infty}\sin\vartheta_i}{(\omega-a^0_{-\infty}\alpha M_{-\infty})^2}(-p^i+p^r)=\frac{1}{\rho^0_{+\infty}}\frac{-\beta}{(\omega-a^0_{+\infty}\alpha M_{+\infty})^2}p^t \tag{219}$$

Eliminating the reflected wave component p^r from (218) and (219) and substituting ω from (201) as well as α from (203) finally yields the amplitude of the transmitted wave in relation to the amplitude of the incident wave:

$$\frac{p^t}{p^i}=\frac{2\sin\vartheta_i\Big[a^0_{-\infty}+\Big(u^0_{-\infty}-u^0_{+\infty}\Big)\cos\vartheta_i\Big]^2}{\sin\vartheta_i\Big[a^0_{-\infty}+\Big(u^0_{-\infty}-u^0_{+\infty}\Big)\cos\vartheta_i\Big]^2+[H(\sigma)-i(1-H(\sigma))]\sqrt{|\sigma|}(a^0_{+\infty})^2} \tag{220}$$

with σ from (211). Note, that for $\sigma<0$ the constant $|p^t|$ represents only the amplitude of the pressure signal at the shear layer position $z=0$. For increasing values of z the amplitude dies out. On the other hand for $\sigma>0$ the amplitude is $|p^t|$ for all $z>0$. For $\sigma=0$ the amplitude of the transmitted wave according to (220) is $|p^t|=2|p^i|$. The angle ϑ_i^0, at which the change of the type of solution takes place corresponds to the angle of perfect reflection (214).

As an example let us consider a shear layer with $M_{-\infty}=M$, $M_{+\infty}=0$, $a^0_{+\infty}=a^0_{-\infty}$. The amplitude ratio of transmitted to incident wave along with the phase jump occuring for angles

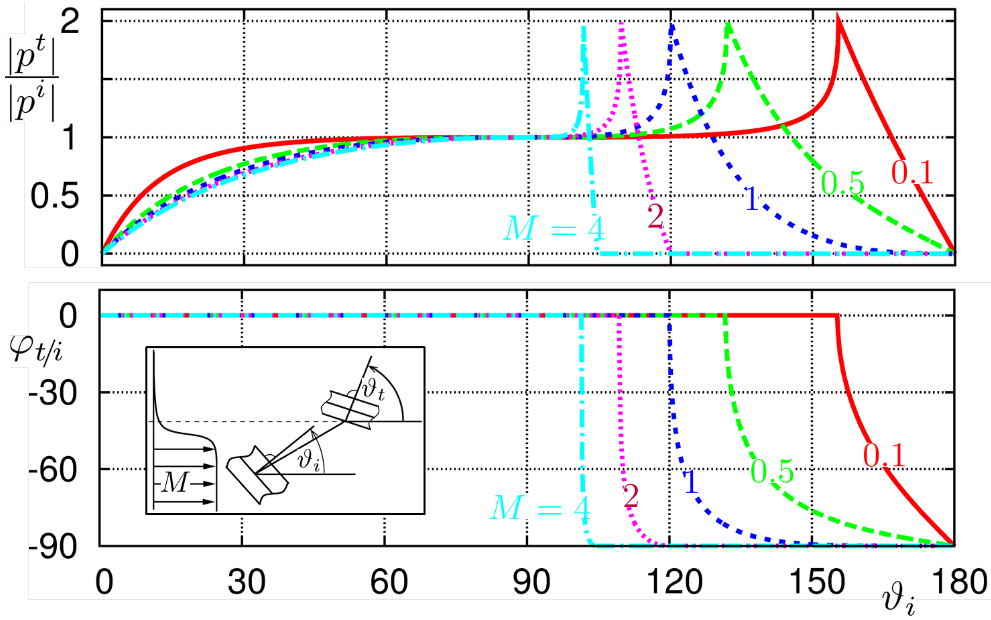

Figure 37: Amplitude ratio and phase jump of transmitted and incident plane wave at shearlayers of different Mach number $M_{-\infty}$ due to (220). Curves for $\vartheta_i > \vartheta_i^0 (= \vartheta_i(|p^t|/|p^i| = 2))$ correspond to pressure signal with exponential decay for $z > 0$. The phase difference $\varphi_{t/i}$ of transmitted to incident wave at shearlayer $z = 0$ corresponds to $p^t/|p^t| = \exp(i\varphi_{t/i})\, p^i/|p^i|$.

above the critical angle of perfect reflection for various Mach numbers M is shown in figure 37. For no flow we would of course obtain a constant $|p^t/p^i| = 1$, since no refraction would occur in this case. Already at small flow Mach numbers (see curve for $M = 0.1$) strong refraction effects are seen for waves hitting the interface at shallow angles. Note that for the wave defining the zone of silence ($\vartheta_i = 0$) the amplitude of the transmitted wave is zero and increases gradually for increasing angles of incidence $\vartheta_i = 0$. For supersonic flows the amplitide ratio was set equal to zero whenever the incidence angle $\vartheta_i > \vartheta_i^c = \arccos(-1/M)$ because for theses angles waves cannot propagate against the flow (note this angle correponds to the Mach cone angle!).

Let us have a look at the reflected part of the wave. This is easily determined using (218) and (220).

$$\frac{p^r}{p^i} = \frac{\sin\vartheta_i[a^0_{-\infty} + (u^0_{-\infty} - u^0_{+\infty})\cos\vartheta_i]^2 - [H(\sigma) - i(1 - H(\sigma))]\sqrt{|\sigma|}(a^0_{+\infty})^2}{\sin\vartheta_i[a^0_{-\infty} + (u^0_{-\infty} - u^0_{+\infty})\cos\vartheta_i]^2 + [H(\sigma) - i(1 - H(\sigma))]\sqrt{|\sigma|}(a^0_{+\infty})^2} \tag{221}$$

Clearly, at perfect reflection ($\sigma = 0$) the incident and the reflected wave have the same amplitude. But what happens for angles beyond perfect reflection? Then $\sigma < 0$ and numerator and denominator become conjugate complex. Since the magnitude of conjugate complex numbers is the same this means that $|p^r/p^i|$ still stays equal to one. However, p^r and p^i are no longer in phase and there occurs a phase shift at the interface. This phase shift happens to be exactly twice the phase jump from the incident signal p^i and the transmitted signal p^t displayed in the lower part of fig. 37.

Finally let us have a look at the apperance of the pressure field at $M = 0.5$ for the considered

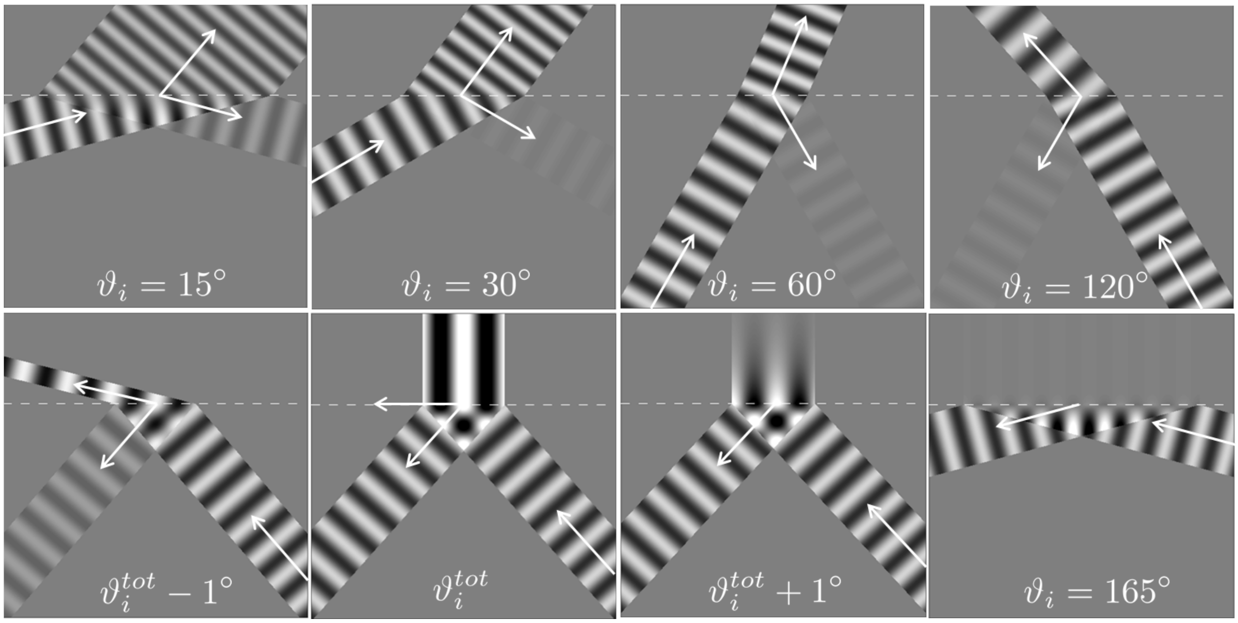

Figure 38: Vizualisation of instantaneous pressure field of plane wave at different incidence angles, $M = 0.5$. Note, for incidence angles beyond angle of specular reflexion, the pressure amplitude is decaying exponentially with distance from shearlayer and the pressure signal is moving with a speed below speed of sound (nearfield).

arrangement of the shearlayer at $M = 0.5$. For illustrational purpose, the complete pressure field is shown on a finite strip of the plane wave only.

Note, how the pressure field above the shearlayer decays exponentially for incidence angles beyond ϑ_i^{tot}. Here the pressure signal moves slower than the speed of sound and therefore is by nature no sound but a nearfield.

If we are only interested in the lower part of the solution ($z \leq 0$) then it is useful to determine the impedance $z_{-\infty}$ of the shearlayer. Since we had expressed the normal particle velocity $\hat{v}_z^-$ already in the expression before (219) we obtain:

$$z_{-\infty} := \frac{\hat{p}^-(z=0)}{\hat{v}_z^-(z=0)} = \frac{\left[a_{-\infty}^0 + \left(u_{-\infty}^0 - u_{+\infty}^0\right)\cos\vartheta_i\right]^2}{\sqrt{\sigma}(a_{+\infty}^0)^2}\, z_{-\infty}^0 \quad , \quad z_{-\infty}^0 = \rho_{-\infty}^0 a_{-\infty}^0 \qquad (222)$$

Let us again consider the case of perfect reflection at $\sigma = 0$. This obviously corresponds to an infinite impedance, just as for the case of an acoustically hard wall! Note that whenever the speed of sound is different below and above the shearlayer, a reflection occurs, even at normal incidence ($\vartheta_i = 90°$).

3.1.3.2 Refraction effects In this section we give some practical examples of sound refraction phenomena.

(a) Aft noise of a jet engine. The jet of a typical turbofan aeroengine is usually hot and at the edge of the jet a shearlayer forms as indicated in the sketch *39*. According to the refraction mechanisms discussed in the previous section, the temperature and the shear layer are arranged as to support the occurence of zones of silence, which here is rather a "cone of silence" downstream of the engine. The noise, which is either generated inside the turboengine (and radiated out of the nozzle exit) or the jet noise itself (which is generated aft of the nozzle exit) has to cross the

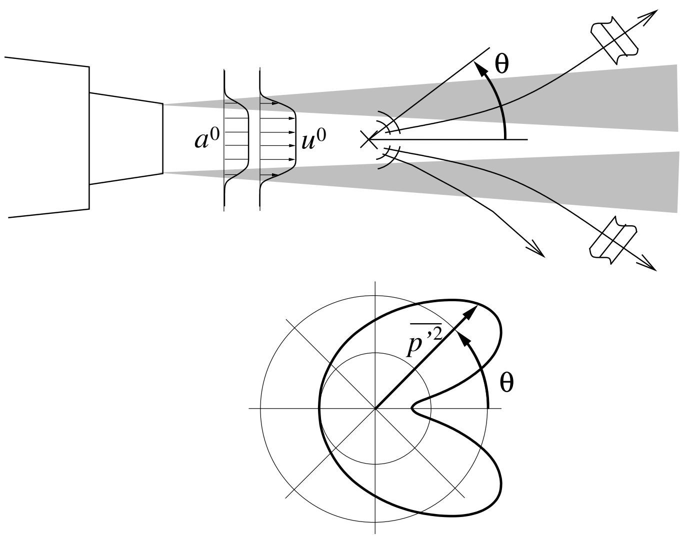

Figure 39: Refraction effect for aft jet engine noise radiation. Occurence of a cone of silence.

shear- and temperature layers of the jet. The refraction leads to a distinct zone of silence for small angles ϑ as shown in the figure. Therefore the observer experiences almost no noise when situated right behind the engines, while at an aft-sideline position the noise is maximum (see sketch of directivity plot in figure *39*).

(b) Intake flow of an turbofan aeroengine. The fan of an aeroengine is ducted by a nacelle. The

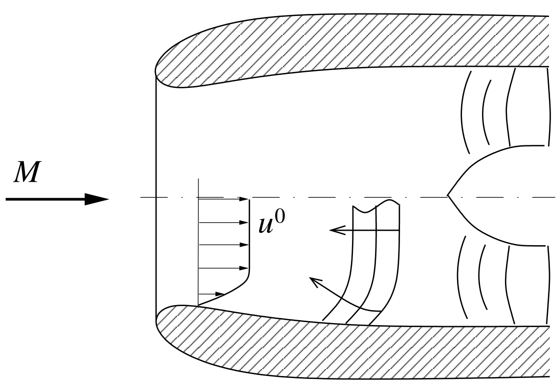

Figure 40: Refraction of turbomachinery noise at intake boundary layer gradients.

noise generated by the rotating fan and the other components of the turbomachinery is propagated against the intake flow. Obviously boundary layers form at the duct surface of the nacelle. According to the discussion in the previous section the flow speed gradients of these boundary layers tend to refract the sound away from the surfaces (see fig *40*). This effect is unfavorable since the sound which otherwise could be absorbed by surface mounted liners, tends to get re-directed towards the centreline of the duct. As a consequence the effectivity of liners is reduced.

(c) Bypass- and core duct flow of aeroengine. The turbomachinery noise, generated inside an aeroengine is partly propagated through the exit ducts. Contrary to the effect in the engine intake the boundary layers act as to refract the sound towards the surface of the duct. Noise absorption by wall mounted liners is here potentially very effective because the turbomachinery sound is

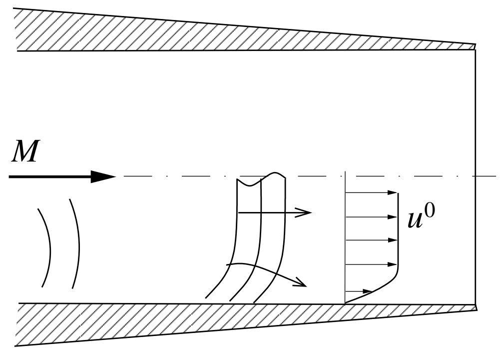

Figure 41: Refraction of turbomachinery noise at exit duct boundary layer gradients.

refracted towards these liners (see fig *41*). The technological difficulty consists in the operation of such liners in a hot stream.

(d) Atmospheric sound propagation at inversion. In stable conditions the temperature of the air in the atmosphere decreases with height. Under certain metereological conditions a so called inversion situation may occur. In such a case at some distinct height there exists a temperature layer within which the temperature rises with height. In such case the environment appears to be more

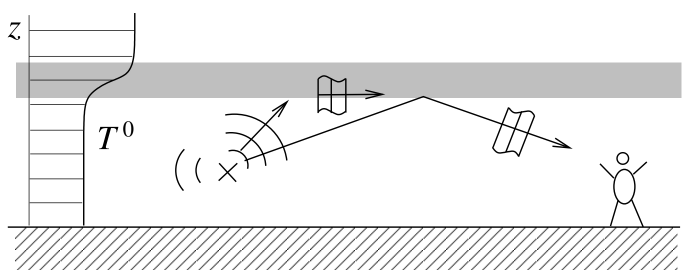

Figure 42: Sound propagation in atmospheric inversion conditions.

noisy as in stable atmospheric conditions. The reason is a perfect reflection at the temperature layer of all the noise generated near the ground. As can be seen from sketch *42* part of the sound which under normal conditions would be radiated up into the sky gets reflected at the layer and is thrown back onto the ground.

(e) Sound propagation in wind. Under appropriate metereological wind conditions noise may be propagated much farther as it would be expected with no wind. Such an increase in noise is experienced when the sound sources are located upstream of the observer. For a source located downstream of the observer, the sound intensity appear strongly reduced or may even disappear completely. The phenomena involved are not primarily connected with the presence of the wind itself, but the gradient in the planetary boundary layer profile. As discussed in the previous section such flow gradients change the propagation characteristics of sound considerably. For the observer located downstream of a source, say a motorway, part of the noise radiated towards the sky is reflected at the boundary layer gradient and re-directed towards the ground (see fig *43*). In this way a so called *wave guide* (German: "Wellenleiter") is formed, which allows for extremely increased noise propagation distances when compared to no-wind conditions. Due to the conditions upstream of the source the observer experiences a zone of silence and noise levels appear highly reduced.

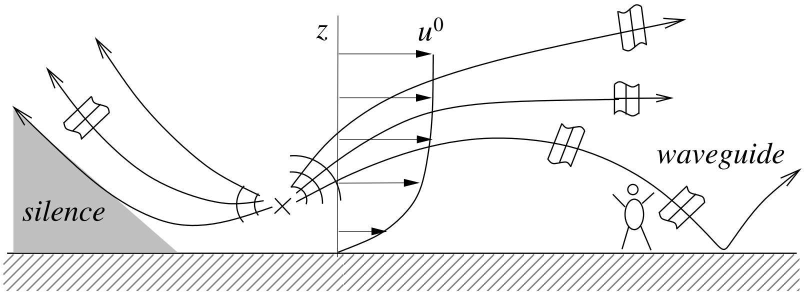

Figure 43: Sound propagation under wind conditions.

3.2 Sound propagation in steady potential flows

In some sense the ”opposite” of parallel shear flows are potential flows as the non trivial extension to a uniform flow field. Since a large set of applied problems in aerodynamics is dealing with attached flows, it has been common to describe such flows as a combination of potential (vorticity free) flows and the (quasi-parallel) viscous layers (e.g. bounday layers). Therefore it is resonable to look at acoustic quantities and sound wave dynamics in potential flows.

3.2.1 Generalization of sound intensity for potential flows

For a medium at rest we were able to define a practically most important conservation quantity for sound, namely the sound power (12). The sound power is determined from the sound intensity, which therefore represents the quantity of main interest. As we saw in section 2.4.1, the sound power is indeed a conservation quantity which is therefore useful to characterize sources or to set up balances. In this section we derive the sound intensity for potential flows.

We start with the linearized momentum balance (56).

$$\frac{\partial \boldsymbol{v}'}{\partial t} + \boldsymbol{v}^0 \cdot \boldsymbol{\nabla} \boldsymbol{v}' + \boldsymbol{v}' \cdot \boldsymbol{\nabla} \boldsymbol{v}^0 + \frac{1}{\rho^0} \boldsymbol{\nabla} p' + \frac{\rho'}{\rho^0} \underbrace{\boldsymbol{v}^0 \cdot \boldsymbol{\nabla} \boldsymbol{v}^0}_{(52) := -\frac{1}{\rho^0} \boldsymbol{\nabla} p^0} = \boldsymbol{0}$$

The potential mean flow is adiabatic, i.e. $p^0/(\rho^0)^\gamma = const.$ Therefore we may replace $-\frac{1}{\rho^0}\boldsymbol{\nabla} p^0 = -\frac{a_0^2}{\rho^0}\boldsymbol{\nabla}\rho^0$. Next, we may simplify the expression $\boldsymbol{v}^0 \cdot \boldsymbol{\nabla} \boldsymbol{v}' + \boldsymbol{v}' \cdot \boldsymbol{\nabla} \boldsymbol{v}^0$ for vorticity free flow and perturbations. For this purpose we consider the vector identity $\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{v}^2 + (\boldsymbol{\nabla} \times \boldsymbol{v}) \times \boldsymbol{v}$. Linearized about the mean flow this gives $\boldsymbol{v}^0 \cdot \boldsymbol{\nabla} \boldsymbol{v}' + \boldsymbol{v}' \cdot \boldsymbol{\nabla} \boldsymbol{v}^0 = (\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v})' = (\frac{1}{2}\boldsymbol{\nabla}\boldsymbol{v}^2 + (\boldsymbol{\nabla} \times \boldsymbol{v}) \times \boldsymbol{v})'$. Since we have zero curl of the flow and the particle velocity the only term remaining of the last expression is $\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{v}^2)' = \boldsymbol{\nabla}(\boldsymbol{v}^0 \cdot \boldsymbol{v}')$. We end up with the following form of the perturbation momentum equation for potential flow:

$$\frac{\partial \boldsymbol{v}'}{\partial t} + \boldsymbol{\nabla}(\boldsymbol{v}^0 \cdot \boldsymbol{v}') + \frac{1}{\rho^0}\boldsymbol{\nabla} p' - \frac{a_0^2 \boldsymbol{\nabla}\rho^0}{(\rho^0)^2}\rho' = \boldsymbol{0}$$

We construct an expression for the perturbation kinetic energy upon forming the dot product of this equation with $(\rho^0 \boldsymbol{v}' + \rho' \boldsymbol{v}^0)$.

$$\frac{\partial \frac{1}{2}\rho^0 \boldsymbol{v}'^2}{\partial t} + \frac{\partial \rho' \boldsymbol{v}^0 \boldsymbol{v}'}{\partial t} \overbrace{- \boldsymbol{v}^0 \boldsymbol{v}' \underbrace{\frac{\partial \rho'}{\partial t}}_{(55):\, = -\boldsymbol{\nabla}\cdot(\rho' \boldsymbol{v}^0 + \rho^0 \boldsymbol{v}')} + (\rho^0 \boldsymbol{v}' + \rho' \boldsymbol{v}^0)\cdot \boldsymbol{\nabla}(\boldsymbol{v}^0 \cdot \boldsymbol{v}')}^{\boldsymbol{\nabla}\cdot[\rho^0 \boldsymbol{v}^0 \cdot \boldsymbol{v}'(\boldsymbol{v}' + \frac{\rho'}{\rho^0}\boldsymbol{v}^0)]} + (\boldsymbol{v}' + \frac{\rho'}{\rho^0}\boldsymbol{v}^0)\cdot \boldsymbol{\nabla} p' - \frac{a_0^2 \boldsymbol{\nabla} \rho^0}{\rho^0}(\boldsymbol{v}' + \frac{\rho'}{\rho^0}\boldsymbol{v}^0)\rho' = \boldsymbol{0}$$

Next we multiply the mass balance equation (54) by p'/ρ^0 while respecting isentropy for the perturbation (i.e. $\rho' = p'/a_0^2$) to get

$$\frac{\partial}{\partial t}\Big(\frac{p'^2}{2\rho^0 a_0^2}\Big) + p' \boldsymbol{\nabla}\cdot\Big(\boldsymbol{v}' + \frac{\rho'}{\rho^0}\boldsymbol{v}^0\Big) + \frac{a_0^2}{\rho^0}\rho'\Big(\boldsymbol{v}' + \frac{\rho'}{\rho^0}\boldsymbol{v}^0\Big)\cdot \boldsymbol{\nabla}\rho^0 = 0$$

The addition of these two last equations finally yields:

$$\frac{\partial}{\partial t}\Big(\frac{\rho^0}{2}\boldsymbol{v}'^2 + \rho' \boldsymbol{v}^0 \cdot \boldsymbol{v}' + \frac{p'^2}{2\rho^0 a_0^2}\Big) + \boldsymbol{\nabla}\cdot\Big((\boldsymbol{v}' + \frac{\rho'}{\rho^0}\boldsymbol{v}^0)(p' + \rho^0 \boldsymbol{v}^0 \cdot \boldsymbol{v}')\Big) = 0$$

Upon time averaging this equation the first term vanishes and we have

$$\boldsymbol{\nabla}\cdot\Big(\overline{(\boldsymbol{v}' + \frac{\rho'}{\rho^0}\boldsymbol{v}^0)(p' + \rho^0 \boldsymbol{v}^0 \cdot \boldsymbol{v}')}\Big) = 0 \tag{223}$$

Had we taken into account mass, momentum and heat sources we would have obtained $\overline{Q} = (\boldsymbol{v}^0 \cdot \boldsymbol{v}' + p'/\rho^0)\dot{\theta}' + (\boldsymbol{v}' + \boldsymbol{v}^0 \rho'/\rho^0)\cdot \boldsymbol{f}'$ instead of zero on the right hand side. In analogy to the reasoning in section 2.4.1 we conclude that the sound intensity for a potential flow is:

$$\boldsymbol{I} := \overline{(\boldsymbol{v}' + \frac{\rho'}{\rho^0}\boldsymbol{v}^0)(p' + \rho^0 \boldsymbol{v}^0 \cdot \boldsymbol{v}')} = \overline{(\boldsymbol{v}' + \frac{p'}{\rho^0 a_0^2}\boldsymbol{v}^0)(p' + \rho^0 \boldsymbol{v}^0 \cdot \boldsymbol{v}')} \tag{224}$$

This result was derived by Myers. For a non-moving medium this definition reduces to the classical definition of sound intensity (9). Even for the trivial case of a uniform flow the expression of the sound intensity does not simplify. The sound power of sound sources in a potential flow is as in the classical definition the integration of the sound intensity over a closed surface surrounding the sources (12). Note that we assumed that mean flow and perturbation are free of vorticity. This means that we have made no statement about the conservation of an acoustic quantity in general flow fields. Sound power (and intensity) could get lost in a shear flow due to the conversion of sound into vortices. Likewise, sound power could be generated when sound waves interact with a vortical flow. The same is true for entropic flows.

3.2.2 Acoustic wave equation in potential flow

The starting point for the description of acoustics in a potential flow field is the identification of a relevant quantity. The stagnation enthalpy B represents such a very important quantity for the characterization of compressible potential flows:

$$B := e_t + \frac{p}{\rho} = h + \frac{1}{2}\boldsymbol{v}^2 \tag{225}$$

Note that B represents "Bernoulli's constant" in 1D compressible flow theory. Therefore it may be interesting to derive an equation, which describes the dynamics of B. In order to obtain such an evolution equation for B we first take the energy balance (34) and subtract the mass balance (32), pre-multiplied with e_t to obtain:

$$\rho\frac{De_t}{Dt} + \boldsymbol{\nabla}\boldsymbol{\cdot}(p\boldsymbol{v}) = \dot{\vartheta} + \boldsymbol{v}\cdot\boldsymbol{f} \tag{226}$$

In order to compose B like in (225) we now need to form an equation for p/ρ. On the one hand we rewrite (32), multiplied by p like

$$p\frac{D\rho^{-1}}{Dt} = \frac{p}{\rho}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} - \frac{p}{\rho^2}\dot{m} \tag{227}$$

On the other hand we multiply (44) by a^2/ρ and have

$$\frac{1}{\rho}\frac{Dp}{Dt} = -a^2\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} + \left(1 - \sigma\frac{p}{\rho T}\right)\frac{a^2}{\rho}\dot{m} + \frac{\sigma a^2}{\rho T}\dot{\vartheta} \tag{228}$$

The addition of (227) and (228) then leaves

$$\rho\frac{Dp/\rho}{Dt} = (p - \rho a^2)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} - \frac{1}{\rho}\left(p - \rho a^2 + \sigma\frac{pa^2}{T}\right)\dot{m} + \frac{\sigma a^2}{T}\dot{\vartheta} \tag{229}$$

Further adding this equation and (226) gives:

$$\rho\frac{DB}{Dt} + \underbrace{\boldsymbol{\nabla}\boldsymbol{\cdot}(p\boldsymbol{v}) - (p - \rho a^2)\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}}_{\boldsymbol{v}\cdot\boldsymbol{\nabla}p + \rho a^2\boldsymbol{\nabla}\cdot\boldsymbol{v} \overset{(44)}{=} -\frac{\partial p}{\partial t} + a^2\left(1 - \frac{\sigma p}{\rho T}\right)\dot{m} + \frac{\sigma a^2}{T}\dot{\vartheta}} = -\frac{1}{\rho}\left(p - \rho a^2 + \sigma\frac{pa^2}{T}\right)\dot{m} + \left(1 + \frac{\sigma a^2}{T}\right)\dot{\vartheta} + \boldsymbol{v}\cdot\boldsymbol{f} \tag{230}$$

Substituting the underbraced expression yields the following simple relation between the pressure and the stagnation enthalpy:

$$\rho\frac{DB}{Dt} - \frac{\partial p}{\partial t} = -\frac{p}{\rho}\dot{m} + \dot{\vartheta} + \boldsymbol{v}\cdot\boldsymbol{f} \tag{231}$$

Here the time derivative of the pressure may again be substituted by use of equation (41) with $\delta = dt\,\partial/\partial t$ (global thermodynamic equilibrium assumed), in which case we have

$$\frac{1}{a^2}\frac{\partial p}{\partial t} = \frac{\partial \rho}{\partial t} + \sigma\rho\frac{\partial s}{\partial t} \overset{(32)}{=} -\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{v}) + \dot{m} + \sigma\rho\frac{\partial s}{\partial t} \tag{232}$$

Executing the substitution of the pressure in (231) then yields

$$\rho\frac{DB}{Dt} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{v}) - \sigma\rho\frac{\partial s}{\partial t} = \left(1 - \frac{p}{\rho a^2}\right)\dot{m} + \frac{1}{a^2}\dot{\vartheta} + \frac{1}{a^2}\boldsymbol{v}\cdot\boldsymbol{f} \tag{233}$$

Next, we have to eliminate the term $\rho\boldsymbol{v}$ in this relation and relate it again to B. This is accomplished by reference to to Crocco's form of the momentum equations (37). Using the vector identity $\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v} = \boldsymbol{\nabla}(\frac{1}{2}\boldsymbol{v}^2) + (\boldsymbol{\nabla}\times\boldsymbol{v})\times\boldsymbol{v}$ and –see (39)– replacing the pressure gradient

$\rho^{-1}\boldsymbol{\nabla} p = \boldsymbol{\nabla} h - T\boldsymbol{\nabla} s$ Crocco obtains (note that a homogeneous fluid in thermodynamic equilibrium is assumed):

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho\boldsymbol{\nabla} B + \rho\boldsymbol{\omega}\times\boldsymbol{v} = \rho T\boldsymbol{\nabla} s + \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \boldsymbol{f} \tag{234}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ represents the vorticity vector. As before we will neglect the friction in Crocco's equation. We may replace $\rho\frac{\partial \boldsymbol{v}}{\partial t}$ by

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = \frac{\partial \rho\boldsymbol{v}}{\partial t} - \boldsymbol{v}\underbrace{\frac{\partial \rho}{\partial t}} \tag{235}$$

$$\overset{(41)}{=} \frac{1}{a^2}\frac{\partial p}{\partial t} - \sigma\rho\frac{\partial s}{\partial t} \overset{(231)}{=} \frac{\rho}{a^2}\frac{DB}{Dt} + \frac{p}{\rho a^2}\dot{m} + \frac{1}{a^2}\dot{\vartheta} + \frac{1}{a^2}\boldsymbol{v}\cdot\boldsymbol{f} - \sigma\rho\frac{\partial s}{\partial t}$$

Using this relation in (234) we have

$$\frac{\partial \rho\boldsymbol{v}}{\partial t} - \frac{\rho\boldsymbol{v}}{a^2}\frac{DB}{Dt} + \sigma\rho\boldsymbol{v}\frac{\partial s}{\partial t} + \rho\boldsymbol{\nabla} B + \rho\boldsymbol{\omega}\times\boldsymbol{v} - \rho T\boldsymbol{\nabla} s = \frac{p\boldsymbol{v}}{\rho a^2}\dot{m} - \frac{\boldsymbol{v}}{a^2}\dot{\vartheta} - \boldsymbol{v}\frac{1}{a^2}\boldsymbol{v}\cdot\boldsymbol{f} + \boldsymbol{f} \tag{236}$$

Finally we eliminate $\rho\boldsymbol{v}$ in (233) and (236) by forming $\frac{D}{Dt}$(233) $-\ \boldsymbol{\nabla}\cdot$(236):

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big[\frac{\rho}{a^2}\frac{DB}{Dt}\Big] + \boldsymbol{\nabla}\cdot\Big[\frac{\rho\boldsymbol{v}}{a^2}\frac{DB}{Dt}\Big] - \boldsymbol{\nabla}\cdot(\rho\boldsymbol{\nabla} B) - \\
&\quad -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\omega}\times\boldsymbol{v}) + \boldsymbol{\nabla}\cdot\Big(\rho T\boldsymbol{\nabla} s - \sigma\rho\boldsymbol{v}\frac{\partial s}{\partial t}\Big) - \frac{\partial}{\partial t}\Big(\sigma\rho\frac{\partial s}{\partial t}\Big) = \\
&\quad = \frac{\partial}{\partial t}\Big[\Big(1 - \frac{p}{\rho a^2}\Big)\dot{m} + \frac{1}{a^2}\dot{\vartheta} + \frac{1}{a^2}\boldsymbol{v}\cdot\boldsymbol{f}\Big] - \boldsymbol{\nabla}\cdot\boldsymbol{f} + \boldsymbol{\nabla}\cdot\Big[\boldsymbol{v}\Big(-\frac{p}{\rho a^2}\dot{m} + \frac{1}{a^2}\dot{\vartheta} + \frac{1}{a^2}\boldsymbol{v}\cdot\boldsymbol{f}\Big)\Big]
\end{aligned} \tag{237}$$

This is as far as we can get to set up an equation for B. Certainly, (237) is not yet solvable, since several unknowns like $\boldsymbol{v}$ or ρ still appear on the left hand side. Therefore let us first of all check, whether some terms will vanish for the case we are interested in, namely potential flow. The vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ of potential flow $\boldsymbol{v}$ is by definition zero, i.e. the first term on the second line in (237) drops out.

As before we split our flow variables into a steady mean flow part $\boldsymbol{v}^0, B^0, \rho^0, s^0$ with zero sources $\dot{m}^0 = 0$, $\boldsymbol{f}^0 = \boldsymbol{0}$, $\dot{\vartheta}^0 = 0$, and a small perturbation about that mean flow $\boldsymbol{v}', B', \rho', s'$ due to some small sources $\dot{m}'$, $\boldsymbol{f}'$, $\dot{\vartheta}'$. First, observe that for a steady potential flow Crocco's equation (234) tells us that

$$\boldsymbol{\nabla} B^0 = \rho^0 T^0\boldsymbol{\nabla} s^0,$$

i.e. the gradient of B^0 is parallel to the gradient of s^0. On the other hand the entropy equation for steady potential flow (40) tells us, that $\boldsymbol{v}^0\cdot\boldsymbol{\nabla} s^0 = 0$, i.e. that there is no change in entropy along a streamline. For constant entropy freestream conditions, this constancy of the entropy is conserved over the whole flow field. This is called *homentropic flow* (German "homentropische Strömung"). From now on we will assume a homentropic steady mean flow. Then the stagnation enthalpy B^0 of the potential flow field is constant (in fact Bernoulli's constant), which is a special feature of potential flows. This is somewhat analogous to parallel flows, featuring a constant mean flow pressure p^0. For such parallel flows we derived equations for the perturbation of the constant, i.e. for p'. For the potential flows in turn, it may be worthwhile trying to derive an equation for B'. We obtain that equation by linearization of (237) about the isentropic, steady potential flow $\boldsymbol{v}^0, B^0, \rho^0$, which yields:

$$\frac{\partial}{\partial t}\Big[\frac{\rho^0}{a_0^2}\frac{D^0 B'}{Dt}\Big] + \boldsymbol{\nabla}\boldsymbol{\cdot}\Big[\frac{\rho^0 \boldsymbol{v}^0}{a_0^2}\frac{D^0 B'}{Dt}\Big] - \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho^0 \boldsymbol{\nabla} B') - \boldsymbol{\nabla}\boldsymbol{\cdot}\Big(\rho^0 T^0 \boldsymbol{\nabla} s' - \sigma^0 \rho^0 \boldsymbol{v}^0 \frac{\partial s'}{\partial t}\Big) - \frac{\partial}{\partial t}\Big(\sigma^0 \rho^0 \frac{\partial s'}{\partial t}\Big) = \tag{238}$$

$$= \underbrace{\frac{\partial}{\partial t}\Big[\Big(1 - \frac{p^0}{\rho^0 a_0^2}\Big)\dot{m}' + \frac{1}{a_0^2}\dot{\vartheta}' + \frac{1}{a_0^2}\boldsymbol{v^0}\cdot\boldsymbol{f'}\Big] - \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{f}' + \boldsymbol{\nabla}\boldsymbol{\cdot}\Big[\boldsymbol{v}^0\Big(-\frac{p^0}{\rho^0 a_0^2}\dot{m}' + \frac{1}{a_0^2}\dot{\vartheta}' + \frac{1}{a_0^2}\boldsymbol{v^0}\cdot\boldsymbol{f'}\Big)\Big]}_{=:\, Q_B}$$

Note that if we supplement the linearized entropy equation (40)

$$\frac{D^0 s'}{Dt} = \frac{1}{\rho^0 T^0}\Big(-\frac{p^0}{\rho^0}\dot{m}' + \dot{\vartheta}'\Big) \tag{239}$$

then (238) and (239) form a set of two equations for the two unknowns B' and s'. For isentropic perturbations (note: acoustic perturbations are isentropic), $s' = 0$ the system decouples and we have one equation for B':

$$\frac{\partial}{\partial t}\Big[\frac{\rho^0}{a_0^2}\frac{D^0 B'}{Dt}\Big] + \boldsymbol{\nabla}\boldsymbol{\cdot}\Big[\frac{\rho^0 \boldsymbol{v}^0}{a_0^2}\frac{D^0 B'}{Dt}\Big] - \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho^0 \boldsymbol{\nabla} B') = Q_B$$

Using the mass balance equation $\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho^0 \boldsymbol{v}^0) = 0$ we may still simplify our B' equation further to

$$\boxed{\frac{D^0}{Dt}\Big[\frac{1}{a_0^2}\frac{D^0 B'}{Dt}\Big] - \frac{1}{\rho^0}\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho^0 \boldsymbol{\nabla} B') = \frac{1}{\rho^0} Q_B} \tag{240}$$

By comparison with the wave equations we have derived so far, we see that (240) is clearly a wave equation. It describes the dynamics of the perturbation of the stagnation enthalpy B' in any steady potential flow. The link of B' to the pressure perturbation p' is easily found in equation (231), evaluated at positions outside the source domain, where $\frac{\partial p'}{\partial t} = \rho^0 \frac{D^0 B'}{Dt}$. The solution of equation (240) needs to be done numerically, since no general Green's function is known.

3.3 Motion of sound sources

So far we have considered sound propagation and noise generation in simple flow fields. A very common situation is that sources are moving, e.g. an airplane flyover, the passing of a train or a car. It is therefore important to consider also the effect of motion on the sound generation and propagation. The situation is as depicted in figure *44*. The medium between the observer and the source is at rest. Therefore, in order to calculate the sound field we have to solve the simple wave equation (61) for a moving source.

3.3.1 Moving mass- or heat source

We first consider the effect of source motion on the sound field of a point mass- or heat source, i.e. the source term $Q_p = \frac{\partial \dot{\theta}'}{\partial t}$ in (61). The source at time t is located at $\boldsymbol{y}(t)$ and has the velocity

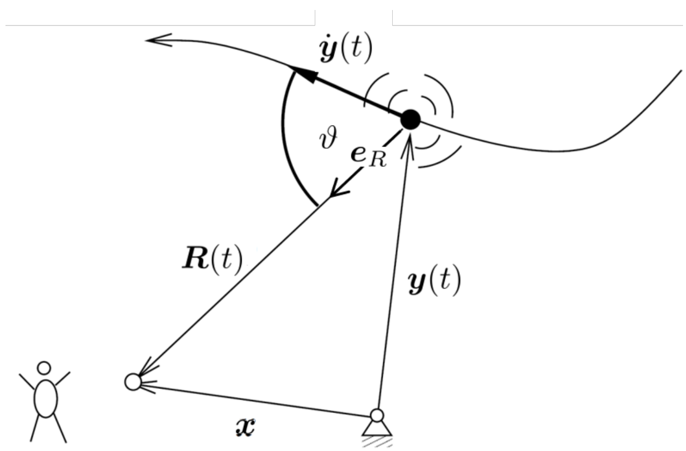

Figure 44: Moving source.

$\dot{\boldsymbol{y}}$. We may therefore describe the source as $\dot{\theta}' = \delta(\boldsymbol{x} - \boldsymbol{y}(t))\theta_p(t)$. We thus have to solve

$$\frac{1}{a_\infty^2}\frac{\partial^2 p'}{\partial t^2} - \Delta p' = \frac{\partial \dot{\theta}'}{\partial t} = \frac{\partial}{\partial t}\Big[\delta(\boldsymbol{x} - \boldsymbol{y}(t))\theta_p(t)\Big]$$

For simplicity of the computation we first determine the velocity potential φ, of which the pressure will be easily derivable:

$$p' = -\rho_\infty \frac{\partial \varphi}{\partial t}$$

The equation to be solved for φ is then

$$\frac{1}{a_\infty^2}\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = -\frac{1}{\rho_\infty}\dot{\theta}' = -\frac{1}{\rho_\infty}\delta(\boldsymbol{x} - \boldsymbol{y}(t))\theta_p(t)$$

We use again the Green's function method to solve the problem. In other words, we express the source term in the source time τ and the source location $\boldsymbol{\xi}$, multiply by the known (freefield) Green's function and integrate over all time and space:

$$\varphi(\boldsymbol{x}, t) = -\frac{1}{\rho_\infty}\int\limits_{-\infty}^{\infty}\int\limits_{V_\infty} \theta_p(\tau)\delta(\boldsymbol{\xi} - \boldsymbol{y}(\tau))\frac{\delta(\tau - t + |\boldsymbol{x} - \boldsymbol{\xi}|/a_\infty)}{4\pi|\boldsymbol{x} - \boldsymbol{\xi}|}\, dV(\boldsymbol{\xi})\, d\tau$$

The volume integral is easily solved and we have

$$\varphi(\boldsymbol{x}, t) = -\frac{1}{4\pi\rho_\infty}\int\limits_{-\infty}^{\infty}\frac{\theta_p(\tau)}{|\boldsymbol{x} - \boldsymbol{y}(\tau)|}\delta(\underbrace{\tau - t + |\boldsymbol{x} - \boldsymbol{y}(\tau)|/a_\infty}_{= g(\tau)})\, d\tau$$

To solve the time integral we use rule (114) and obtain immediately

$$\varphi(\boldsymbol{x}, t) = -\frac{1}{4\pi\rho_\infty}\sum_{i=1}^{n}\frac{\theta_p(\tau_i)}{|\boldsymbol{x} - \boldsymbol{y}(\tau_i)|}\frac{1}{\left|\frac{dg}{d\tau}\right|_{\tau_i}}$$

where τ_i denotes the i^{th} zero of $g(\tau)$, i.e.

$$g(\tau_i) = 0 = \tau_i - t + |\boldsymbol{x} - \boldsymbol{y}(\tau_i)|/a_\infty \tag{241}$$

We may still express $\frac{dg}{d\tau}$ more explicitly

$$\frac{dg}{d\tau} = 1 - \underbrace{\frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|}}_{\frac{\boldsymbol{R}}{R}=\boldsymbol{e}_R} \cdot \underbrace{\frac{d\boldsymbol{y}}{d\tau}\frac{1}{a_\infty}}_{\boldsymbol{M}_q}$$

where we have introduced the (vectorial) Mach number of the source motion $\boldsymbol{M}_q$, the distance vector $\boldsymbol{R}$ between source and observer. Finally we have for the i^{th} contribution φ_i to φ

$$\varphi_i(\boldsymbol{x},t) = -\frac{1}{4\pi\rho_\infty}\frac{\theta_p(\tau_i)}{R_i|1-\boldsymbol{e}_R\cdot\boldsymbol{M}_q|} \quad , R_i = |\boldsymbol{x}-\boldsymbol{y}(\tau_i)|$$

We find the pressure from the definition of the velocity potential to be

$$p_i'(\boldsymbol{x},t) = -\rho_\infty\frac{\partial\varphi_i}{\partial t} = \frac{1}{4\pi}\underbrace{\frac{\partial}{\partial t}\Big[\frac{\theta_p(\tau_i)}{R_i|1-\boldsymbol{e}_R\cdot\boldsymbol{M}_q|}\Big]}_{\frac{\frac{\partial\theta_p}{\partial t}}{R_i|1-\boldsymbol{e}_R\cdot\boldsymbol{M}_q|} - \theta_p\frac{\frac{\partial}{\partial t}\left(R_i-\boldsymbol{R}_i\cdot\boldsymbol{M}_q\right)}{R_i^2(1-\boldsymbol{e}_R\cdot\boldsymbol{M}_q)^2}}$$

The second term in the underbrace can be expressed more explicitly

$$\frac{\partial}{\partial t}\left(R_i-\boldsymbol{R}_i\cdot\boldsymbol{M}_q\right) = \frac{\partial R_i}{\partial t} - \frac{\partial\boldsymbol{M}_q}{\partial t}\cdot\boldsymbol{R}_i - \boldsymbol{M}_q\cdot\frac{\partial\boldsymbol{R}_i}{\partial t}$$

Now $R_i = a_\infty(t-\tau_i)$ and therefore $\frac{\partial R_i}{\partial t} = a_\infty\left(1-\frac{d\tau_i}{dt}\right)$. We obtain $\frac{d\tau_i}{dt}$ by differentiation of (241) with respect to t:

$$\frac{d\tau_i}{dt} - 1 - \boldsymbol{e}_R\cdot\boldsymbol{M}_q\frac{d\tau_i}{dt} = 0 \quad \Longrightarrow \frac{d\tau_i}{dt} = \frac{1}{1-\boldsymbol{e}_R\cdot\boldsymbol{M}_q}$$

Now we may express $\frac{\partial R_i}{\partial t} = -\frac{\boldsymbol{e}_R\cdot\boldsymbol{M}_q a_\infty}{1-\boldsymbol{e}_R\cdot\boldsymbol{M}_q}$, while $\frac{\partial\boldsymbol{R}_i}{\partial t} = \frac{\partial}{\partial t}(\boldsymbol{x}-\boldsymbol{y}(\tau_i)) = -a_\infty\frac{\boldsymbol{M}_q}{1-\boldsymbol{e}_R\cdot\boldsymbol{M}_q}$.

Finally we use $\frac{\partial\theta_p}{\partial t} = \frac{\partial\theta_p}{\partial\tau}\frac{d\tau}{dt}$ and after collection of all terms we obtain as solution for the pressure

$$p_i'(\boldsymbol{x},t) = \frac{1}{4\pi}\left\{\frac{\frac{\partial\theta_p}{\partial\tau} + \boldsymbol{e}_R\cdot\frac{\partial\boldsymbol{M}_q}{\partial\tau}(1-M_{qR})^{-1}\theta_p}{R_i(1-M_{qR})^2} + \frac{a_\infty(M_{qR}-\boldsymbol{M}_q^2)}{R_i^2(1-M_{qR})^3}\theta_p\right\} \tag{242}$$

where $M_{qR} = \boldsymbol{e}_R\cdot\boldsymbol{M}_q$ is the momentary Mach number component in the direction of the observer, which may alternatively also be expressed as $M_{qR} = M_q\cos\theta$, where θ denotes the angle under which the observer sees the source according to figure *44*.

The pressure field of the moving mass- or heat source displays several distinct features, which are not observed for a non-moving source:

- contrary to the case of a non-moving source the pressure contains a nearfield (second term in (242)) proportional to R_i^{-2}; the farfield is proportional to R_i^{-1}, which is the only relevant contribution far away from the source. This part represents the sound field.
- When compared to a non-moving source of the same strength θ_p a convective amplification occurs, such that the signal strength is amplified by a factor of $(1 - M_{qR})^{-2}$ in the direction of the motion, while a signal reduction of $(1 + M_{qR})^{-2}$ occurs against the direction of the motion.
- Contrary to the case of a non-moving source of the same strength θ_p a sound field exists also for a steady source $\dfrac{\partial \theta_p}{\partial \tau} = 0$ as long as the motion is accelerated, i.e. $\dfrac{\partial \boldsymbol{M}_q}{\partial \tau} \neq 0$. This has far reaching consequences. From potential flow theory it is known that some object in a flow may be represented by a source/sink distribution as our assumed mass sources. If e.g. a propeller blade is represented by such a distribution of mass sources equation (242) shows that the propeller blade generates noise because of its accelerated motion.

For constant source speed the change in the pressure field due to the motion of the source may be compared to the results we obtained for a mass- or heat source in a steady flow eqn (176). It is seen that the amplification factors are quite different even for the same Mach number. Only the correct representation of the different amplification functions allows to transfer results obtained in a windtunnel measurement to real flyover situations.

3.3.2 Moving point force

If we take as a source a given point source, in which case the equation

$$\frac{1}{a_\infty^2}\frac{\partial^2 p'}{\partial t^2} - \Delta p' = -\boldsymbol{\nabla}\boldsymbol{\cdot}\Big[\delta(\boldsymbol{x} - \boldsymbol{y}(t))\boldsymbol{f}_p(t)\Big]$$

is to be solved, an analogous derivation as for the mass- or heat source leads to the following result

$$p_i'(\boldsymbol{x}, t) = \frac{1}{4\pi}\left\{\frac{\dfrac{\partial \boldsymbol{f}_p}{\partial \tau}\boldsymbol{\cdot}\boldsymbol{e}_R + (\boldsymbol{f}_p\boldsymbol{\cdot}\boldsymbol{e}_R)\dfrac{\partial \boldsymbol{M}_q}{\partial \tau}\boldsymbol{\cdot}\boldsymbol{e}_R(1 - M_{qR})^{-1}}{a_\infty R_i(1 - M_{qR})^2} + \frac{-\boldsymbol{f}_p\boldsymbol{\cdot}\boldsymbol{M}_q + (1 - \boldsymbol{M}_q^2)\boldsymbol{f}_p\boldsymbol{\cdot}\boldsymbol{e}_R(1 - M_{qR})^{-1}}{R_i^2(1 - M_{qR})^2}\right\} \tag{243}$$

For constant source speed the change in the pressure field due to the motion of the source may be compared to the results we obtained for a force point source in a steady flow eqn (178). It is seen that the amplification factors are quite different even for the same Mach number. Only the correct representation of the different amplification functions allows to transfer results obtained in a windtunnel measurement to real flyover situations.

3.3.3 Moving multipole

In section 2.6.4 we discussed the expansion of general source terms into more simple components (superposition of poles of various orders). Here we have a look at the way in which a subsonic motion of the source changes the multipole contributions. An expansion of the same kind as in 2.6.4 for (144) and (145) leads to terms (pole of order m):

$$p'_{ijk}(\boldsymbol{x},t) = \frac{1}{4\pi}(-1)^{i+j+k}\frac{\partial^{i+j+k}}{\partial x_1^i \partial x_2^j \partial x_3^k}\left[\frac{m_{ijk}(\tau_0)}{R_0|1-M_{qR_0}|}\right], \quad m = i+j+k \tag{244}$$

with $\tau = t - R(\tau)/a_\infty$ and $M_{qR} = \boldsymbol{e}_R \cdot \boldsymbol{M}_q$ as in the previous section. In the farfield only contributions proportional to $1/R$ exist, i.e. for subsonic motion

$$p'_{ijk}(\boldsymbol{x},t) \simeq \frac{1}{4\pi}(-1)^{i+j+k}\frac{1}{R_0}\frac{\partial^{i+j+k}}{\partial x_1^i \partial x_2^j \partial x_3^k}\left[\frac{m_{ijk}(\tau_0)}{1-M_{qR_0}}\right], \quad m = i+j+k$$

Since the relative Machnumber of the motion w.r.t. the observer depends on R (and therefore x) each derivative with respect to $x_{1,2,3}$ of $m_{ijk}(\tau_0)$ produces (through the retarded time τ_0) the convection factor $(1 - M_{qR})$ and in the end higher order terms like $(1 - M_{qR})^{-(m+1)}$. In other words, as a consequence of the source motion a convective amplifiction occurs for each multipole contribution differently. The higher order the multipole moment is, the stronger is the convective amplification (beaming in flow direction).

3.3.4 Doppler effect

We consider a moving monopole source producing the field

$$p'(\boldsymbol{x},t) = \frac{1}{4\pi}\frac{f(i\omega_0\tau)}{R|1-M_{qR}|}$$

where $\omega_0 = const$ describes the time dependence of the source travelling along with the source and f is the signal shape and M_{qR} denotes again the Mach number component towards the observer. The phase of the field is

$$\Phi = \omega_0\tau$$

Now, generally the frequency is defined as the time derivative of the phase. For the sender this is

$$\omega_s = \frac{\partial\Phi}{\partial\tau} = \omega_0$$

while for the observer this is:

$$\omega_r = \frac{\partial\Phi}{\partial t} = \omega_0\frac{d\tau}{dt} = \frac{\omega_0}{1-M_{qR}}$$

which means that the observer receives a frequency, which is different from the one sent out. It is interesting to note that the frequency shift is according to the convection factor $\frac{d\tau}{dt} = \frac{1}{|1-M_{qR}|}$, which was already responsible for the convective amplification. The frequency shift is called *Doppler shift*. Specifically one obtains an increase in the frequency if $M_{qR} > 0$ (component of source motion towards observer), while a decrease in frequency is observed for $M_{qR} < 0$ (component of source motion away from observer).

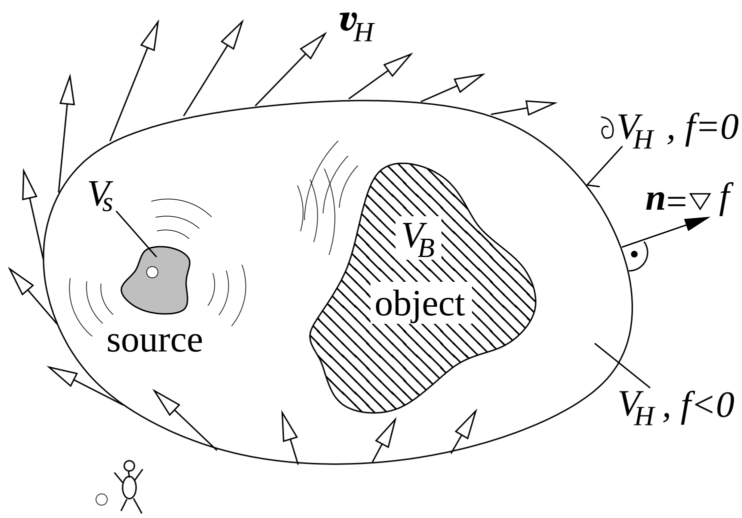

Figure 45: Kirchhoff integration surface ∂V_H.

3.3.5 Kirchhoff integral for moving surface

If the pressure and its derivatives are known on an arbitrarily shaped closed (generally moving) surface ∂V_H (see figure 45), enclosing all source domains V_S and all objects V_B, then the sound pressure field on all locations outside ∂V_H may be determined with the Kirchhoff integral.

The Kirchhoff integral may be derived in a quite similar way as (130) or (131).

We define the domain V_H inside of ∂V_H with the help of the scalar function $f(\boldsymbol{x}, t)$, whose zero-level surface is identical with ∂V_H:

$$\begin{array}{lll} f(\boldsymbol{x},t) < 0 & & \boldsymbol{x} \in V_H \\ f(\boldsymbol{x},t) = 0 & \text{for} & \boldsymbol{x} \in \partial V_H \\ f(\boldsymbol{x},t) > 0 & & \text{else} \end{array} \tag{245}$$

Also we require by simple normalization that $|\boldsymbol{\nabla} f|_{f=0} = 1$. Now we introduce again a new pressure variable $\underline{p}' = H(f)p'$ and apply the wave operator $\frac{1}{a_\infty^2}\frac{\partial^2}{\partial t^2} - \Delta$ on the right and left hand sides of this definition equation. After elementary application of the product and chain rules of differentiation we obtain

$$\frac{1}{a_\infty^2}\frac{\partial^2 \underline{p}'}{\partial t^2} - \Delta \underline{p}' = \underbrace{HQ_p}_{=0} + \underbrace{\frac{1}{a_\infty^2}\frac{\partial}{\partial t}\left[\delta(f)\frac{\partial f}{\partial t}p'\right]}_{=:\, I_1} + \underbrace{\frac{1}{a_\infty^2}\delta(f)\frac{\partial f}{\partial t}\frac{\partial p'}{\partial t}}_{=:\, I_2} - \underbrace{\boldsymbol{\nabla}\cdot[\delta(f)\boldsymbol{n}p']}_{=:\, I_3} - \underbrace{\delta(f)\boldsymbol{n}\cdot\boldsymbol{\nabla}p'}_{=:\, I_4} \tag{246}$$

where the unit normal vector $\boldsymbol{n} = \boldsymbol{\nabla} f|_{f=0}$ points outside of ∂V_H. We still need to express $\frac{\partial f}{\partial t}$ on the surface. The motion of the surface is defined by the (given) velocity vector $\boldsymbol{v}_H(\boldsymbol{x}, t)$. The location of the boundary, i.e. the isosurface $f = 0$, is found by requiring the total time derivative of f to be zero:

$$\frac{\partial f}{\partial t} + \boldsymbol{v}_H \cdot \underbrace{\boldsymbol{\nabla} f}_{\boldsymbol{n}} = 0 \tag{247}$$

which we may transform to obtain

$$\frac{\partial f}{\partial t} = -\boldsymbol{v}_H \cdot \boldsymbol{n} =: -v_n \quad . \tag{248}$$

We may now solve (247) by the Green's function method, i.e. multiply by the 3D free field Green's function $G_0 = \delta(g)/(4\pi r)$, $g = \tau - t + r/a_\infty$, and integrate over all space V_∞ and all time.

We treat the four contributions in (247) separately:

$$\begin{aligned}
\int I_1 G_0 &= \frac{1}{4\pi a_\infty^2} \int\limits_{V_\infty} \int\limits_{-\infty}^{\infty} \frac{\partial}{\partial \tau}\Big[\delta(f)(-v_n)p'\Big]\frac{\delta(g)}{r} d\tau\, dV(\xi) \\
&= \frac{-1}{4\pi a_\infty} \int\limits_{V_\infty} \Big[\delta(f) M_n p' \underbrace{\frac{\delta(g)}{r}}_{=0}\Big]_{-\infty}^{\infty} dV(\xi) + \frac{1}{4\pi a_\infty^2} \int\limits_{-\infty}^{\infty} \int\limits_{V_\infty} \delta(f) v_n \frac{p'}{r} \underbrace{\frac{\partial \delta(g)}{\partial \tau}}_{-\partial\delta/\partial t} dV(\xi)\, d\tau \\
&= -\frac{1}{4\pi}\frac{\partial}{\partial t}\Big[\int\limits_{\partial V_H} \frac{M_n}{a_\infty}\frac{p'}{r}\, dS(\xi)\Big]_{\tau = t - r/a_\infty}
\end{aligned}$$

where we have introduced the acoustic Mach number $M_n = v_n/a_\infty$ of the surface normal component of the surface velocity.

$$\int I_2 G_0 = \frac{1}{4\pi a_\infty^2} \int\limits_{-\infty}^{\infty} \int\limits_{V_\infty} \delta(f)(-v_n)\frac{\partial p'}{\partial \tau}\frac{\delta(g)}{r}\, dV(\xi)\, d\tau = -\frac{1}{4\pi}\Big[\int\limits_{\partial V_H} \frac{M_n}{a_\infty}\frac{\partial p'}{\partial \tau}\frac{1}{r}\, dS(\xi)\Big]_{\tau = t - r/a_\infty}$$

$$\begin{aligned}
\int I_3 G_0 &= -\frac{1}{4\pi} \int\limits_{\partial V_\infty} \int\limits_{-\infty}^{\infty} \underbrace{\delta(f)}_{=0} \boldsymbol{n} p' \frac{\delta(g)}{r}\, d\tau\, dS(\xi) + \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} \int\limits_{V_\infty} \delta(f)\, p' \frac{\partial}{\partial n}\Big(\frac{\delta(g)}{r}\Big)\, dV(\xi)\, d\tau \\
&= -\frac{1}{4\pi}\frac{\partial}{\partial t}\Big[\int\limits_{\partial V_H} \frac{1}{a_\infty}\frac{p'}{r}\frac{\partial r}{\partial n}\, dS(\xi)\Big]_{\tau = t - r/a_\infty} - \frac{1}{4\pi}\Big[\int\limits_{\partial V_H} \frac{p'}{r^2}\frac{\partial r}{\partial n}\, dS(\xi)\Big]_{\tau = t - r/a_\infty}
\end{aligned}$$

where the last equality follows from using the relation $p'\frac{\partial}{\partial n}\Big(\frac{\delta(g)}{r}\Big) = \Big(p'\frac{d\delta}{dg}\frac{\partial g}{\partial r}\frac{1}{r} - \delta\frac{p'}{r^2}\Big)\frac{\partial r}{\partial n} = \Big(p'\frac{d\delta}{d\tau}\frac{1}{a_\infty r} - \delta\frac{p'}{r^2}\Big)\frac{\partial r}{\partial n} = -\frac{1}{a_\infty}\frac{\partial}{\partial t}\Big(\delta\, p'\frac{\partial r}{\partial n}\Big) - \delta\frac{p'}{r^2}\frac{\partial r}{\partial n}$ in the volume integral. Note, that we have used the notation $\frac{\partial}{\partial n} := \boldsymbol{n}\cdot\boldsymbol{\nabla}_\xi$ as directional derivative. The final integral maybe integrated straight forward:

$$\int I_4 G_0 = -\frac{1}{4\pi}\Big[\int\limits_{\partial V_H} \frac{1}{r}\frac{\partial p'}{\partial n}\, dS(\xi)\Big]_{\tau = t - r/a_\infty}$$

Upon adding all four integrals we obtain the pressure $\underline{p'}$ at any position outside the Kirchhoff surface to be

$$\underline{p}'(\boldsymbol{x},t) = -\frac{1}{4\pi}\Big[\int\limits_{\partial V_H} \frac{1}{r}\Big\{\frac{M_n}{a_\infty}\frac{\partial p'}{\partial \tau} + \frac{\partial p'}{\partial n} - \frac{p'}{r}\,\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{e}_r\Big\} dS(\xi)\Big]_{\tau=t-r/a_\infty} - \\ -\frac{1}{4\pi a_\infty}\frac{\partial}{\partial t}\Big[\int\limits_{\partial V_H} \frac{p'}{r}(M_n - \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{e}_r)\,dS(\xi)\Big]_{\tau=t-r/a_\infty} \qquad (249)$$

The Kirchhoff integral is written for a closed integration surface in arbitrary (subsonic) motion through a medium at rest. The observer at $\boldsymbol{x}$ is at rest as well. The normal vector $\boldsymbol{n}$ on the surface element $dS(\boldsymbol{\xi})$ is by definition pointing towards the exterior of the surface; the unit vector from source element to observer is $\boldsymbol{e}_r = (\boldsymbol{x} - \boldsymbol{\xi})/r$. Note, that the integrals need to be evaluated at the retarded time $\tau = t - r/a_\infty$, which is non-trivial when the surface is moving because then $r = r(\tau)$. If only time derivative data is available in the source time τ, the Kirchoff equation may be re-written as

$$\underline{p}'(\boldsymbol{x},t) = -\frac{1}{4\pi}\Big[\int\limits_{\partial V_H} \frac{1}{r}\Big\{\frac{M_n}{a_\infty}\frac{\partial p'}{\partial \tau} + \frac{\partial p'}{\partial n} - \frac{p'}{r}\,\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{e}_r\Big\} dS(\xi)\Big]_{\tau=t-r/a_\infty} - \\ -\frac{1}{4\pi a_\infty}\Big[\int\limits_{\partial V_H} \Big\{\frac{\partial}{\partial \tau}\Big[\frac{p'}{r}(M_n - \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{e}_r)\Big] + \frac{p'}{r}(M_n - \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{e}_r)\frac{d\dot{S}}{dS}\Big\}\frac{1}{1-\boldsymbol{e}_r\boldsymbol{\cdot}\boldsymbol{M}}\,dS(\xi)\Big]_{\tau=t-r/a_\infty}$$

Here $\dfrac{d\dot{S}}{dS}$ denotes the temporal change in surface element area, when for instance the surface is extending or shrinking; this expression is zero for a rigid body motion. Traditionally the Kirchhoff formula has been used for the prediction of helicopter noise, based on surface data obtained from numerical fluid mechanics (CFD) simulations.

4 Description of aerodynamic sources of sound

This section is devoted to pinning down quantities which represent aeroacoustic sources. It must be admitted that no general answer to the question of describing aerodynamic sources of sound has been found yet. We introduce the concept of aeroacoustic analogies with the most famous Lighthill acoustic analogy. Then we introduce Lilley's and Möhring's analogies as generalizations of Lighthill's analogy. Next Ffowcs-Williams and Hawkings generalization of Lighthill's analogy to explicitly include boundaries is discussed.

4.1 The quiecent-fluid view: Lighthill's acoustic analogy

Knowing about the general form of the acoustic wave operator in a non-moving medium one may try to re-arrange the general balance equations for mass and momentum such that exactly this form is obtained. The derivation is quite simple. Upon taking $\frac{\partial}{\partial t}(32) - \boldsymbol{\nabla}\cdot(33)$ gives:

$$\frac{\partial^2 \rho}{\partial t^2} = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}\boldsymbol{v} + p\boldsymbol{I} - \boldsymbol{\tau}) \tag{250}$$

Next we subtract the term $a_\infty^2 \Delta\rho$ from left- and right hand side and obtain:

$$\frac{\partial^2 \rho}{\partial t^2} - a_\infty^2 \Delta\rho = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\cdot(\underbrace{\rho \boldsymbol{v}\boldsymbol{v} + (p - a_\infty^2\rho)\boldsymbol{I} - \boldsymbol{\tau}}_{\boldsymbol{T}}) \tag{251}$$

The left hand side of (251) now clearly represents the wave operator for a non-moving homogeneous medium with a constant speed of sound of a_∞. The quantity $\boldsymbol{T}$ is called *Lighthill's stress tensor* (German: "Lighthill'scher Spannungstensor").

If we introduce $p' := p - p_\infty$, $\rho' := \rho - \rho_\infty$ as deviations from respective constant reference values of pressure p_∞, density ρ_∞ and speed of sound a_∞ we may as well write (251) like:

$$\frac{\partial^2 \rho'}{\partial t^2} - a_\infty^2 \Delta\rho' = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\cdot(\underbrace{\rho \boldsymbol{v}\boldsymbol{v} + (p' - a_\infty^2\rho')\boldsymbol{I} - \boldsymbol{\tau}}_{\boldsymbol{T}}) \tag{252}$$

This famous wave equation was published by M. Lighthill 1952 and it forms the basis of *Lighthills aeroacoustic analogy* (German "Lighthill'sche aeroakustische Analogie"). Note, that for small deviations from the reference state (∞) the expression $s' := (p' - a_\infty^2\rho')/(\sigma\rho_\infty a_\infty^2)$ represents nothing but the entropy perturbation s'. In the following we discuss some features of this equation.

- compare the left hand side of (252) with the classical wave equation for the pressure (61) for non-moving homogeneous media
- consider the aeroacoustic problem as analogous to an acoustic problem in a ficticious non-moving medium; therefore we call this concept "aeroacoustic analogy".

- Due to the identification of the left hand side as a wave operator, the right hand side represents the aeroacoustic sources
- Three distinct source mechanism may be read from Lighthill's stress tensor $\boldsymbol{T}$:

  1. Changes in flow velocities: $\rho \boldsymbol{v}\boldsymbol{v}$ (e.g. turbulence)
  2. Changes of the entropy $\sim (p' - a_\infty^2 \rho')$ (e.g. temperature fluctuations due to combustion)
  3. changes in the viscous friction stresses $\boldsymbol{\tau}$ (usually considered to be unimportant in most cases)

- The source term $Q_L = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{T}$ represents the double divergence of a tensor. This means that the leading order term of the multipole expansion of Q_L is of the order $2^2 = 4$ or a "quadrupole". The aeroacoustic source terms may therefore be characterized as of "quadrupole type".
- Pros of the analogy concept of Lighthill:

  + all methods from classical acoustics are immediately applicable

  + the solution of the acoustic part of the aeroacoustic problem is simple (partly analytical)

- Cons of the analogy concept of Lighthill:

  - the right hand side must be modelled (or has to be simulated numerically).

  - Sound propagation phenomena (refraction at shear- or boundary layers) appear as sources, although they are obviously only kinematic effects. Lighthill's analogy cannot describe these effects; they would somehow have to be modelled as equivalent sources.

  - the interpretation of the right hand side as "given source term" is questionable in cases, where $\boldsymbol{T}$ depends of the acoustic field, i.e. the solution. This is typical for aeroacoustic feedback or resonance phenomena, e.g. at open shallow cavities in tangential flow.

- properties of Lighthill's analogy equation (252):

  * Equation (252) is an exact consequence of the balance equations of continuum mechanics, i.e. as it stands, no simplification was introduced.

  * ρ' may be interpreted as acoustic signal only in domains where the time averages $\overline{\boldsymbol{v}} = \boldsymbol{0}$, $\overline{p} = p_\infty$, $\overline{\rho} = \rho_\infty$ because of the underlying wave operator. This is indeed the case for a localized aeroacoustic source (e.g. a turbulent jet flow), which radiates sound into the quiecent surrounding air with a mean density of ρ_∞ and mean pressure of p_∞. The same is true for a uniformly moving surrounding medium with $\overline{\boldsymbol{v}} = \boldsymbol{v}_\infty = const$, $\overline{p} = p_\infty$[5]

[5] in this case one may derive a convected wave equation as (161) by substituting $\boldsymbol{v} = \boldsymbol{v}_\infty + \boldsymbol{v}'$ in (252). Parts of the term $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}\boldsymbol{v})$ may then be moved to the left hand side yielding

$$\frac{D_\infty^2 \rho'}{Dt^2} - a_\infty^2 \Delta \rho' = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}' \boldsymbol{v}' + (p' - a_\infty^2 \rho') \boldsymbol{I} - \boldsymbol{\tau}) \tag{253}$$

* The left hand side of (252), namely the wave operator for the medium at rest is "self adjoint"; which means that in the spectral space (Fourier-transformed) it contains only real temporal eigenvalues ω. This means that temporally amplified, e.g. self excited processes without external forcing are excluded. Those would be characterized by complex $\omega = \omega_r + i\omega_i$ with $\omega_i > 0$. Such phenomena may exist e.g. in shear flows, where they are called "hydrodynamic instabililies". Such contributions to the solutions may be quite cumbersome, such that the "self adjointness" of the wave operator is a welcome feature. This property is important and not obvious because other analogy equations were derived, which are not self adjoint and contain unwanted solutions.

Note, that a similar derivation as for (252) can be done for the pressure instead of the density. Then we obtain the pressure form of Lighthill's analogy:

$$\frac{1}{a_\infty^2}\frac{\partial^2 p'}{\partial t^2} - \Delta p' = \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \boldsymbol{v}\boldsymbol{v} - \boldsymbol{\tau}) + \frac{1}{a_\infty^2}\frac{\partial^2}{\partial t^2}(p' - a_\infty^2 \rho') \tag{254}$$

For an isentropic problem (cold flow) the pressure and the density form of Lighthill's inhomogeneous wave equation is virtually the same.

4.1.1 Solution of Lighthill's equation for free field problems

Let us first consider aerodynamic sound generation in unbounded domain. Since Lighthill's acoustic analogy represents a wave equation for a homogeneous medium at rest we may use the free field Green's function to solve for ρ' (see section 2.5.2):

$$\rho'(\boldsymbol{x}, t) = \frac{1}{4\pi a_\infty^2} \int_{-\infty}^{\infty} \int_{V_\infty} \boldsymbol{\nabla}_\xi \boldsymbol{\cdot} (\boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \boldsymbol{T}) \frac{\delta(\tau - t + r/a_\infty)}{r} \, dV(\xi) \, d\tau.$$

The time integral can immediately be solved thanks to the delta function. Then we obtain:

$$\rho'(\boldsymbol{x}, t) = \frac{1}{4\pi a_\infty^2} \int_{V_\infty} \frac{1}{r} \boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \boldsymbol{T}(t - r/a_\infty, \boldsymbol{\xi}) \, dV(\xi) \tag{255}$$

We may re-write this solution by manipulation the second last equation. We re-formulate the integrand by use of the product rule:

$$\rho'(\boldsymbol{x}, t) = \frac{1}{4\pi a_\infty^2} \int_{-\infty}^{\infty} \int_{V_\infty} \boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \Big(\frac{\delta}{r} \boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \boldsymbol{T}\Big) - (\boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \boldsymbol{T}) \boldsymbol{\cdot} \boldsymbol{\nabla}_\xi \Big(\frac{\delta}{r}\Big) \, dV(\xi) \, d\tau$$

The use of Gauss' theorem shows that the first term yields zero since it is reasonably assumed that the sources do not extend out to infinity. The remaining term may again be split by the product rule to yield

$$\rho'(\boldsymbol{x}, t) = -\frac{1}{4\pi a_\infty^2} \int_{-\infty}^{\infty} \int_{V_\infty} \boldsymbol{\nabla}_\xi \boldsymbol{\cdot} \Big(\boldsymbol{T} \boldsymbol{\cdot} \boldsymbol{\nabla}_\xi \Big(\frac{\delta}{r}\Big)\Big) - \boldsymbol{T} \boldsymbol{:} \Big(\underbrace{\boldsymbol{\nabla}_\xi \boldsymbol{\nabla}_\xi \frac{\delta}{r}}_{= \boldsymbol{\nabla}_x \boldsymbol{\nabla}_x \frac{\delta}{r}} \Big) \, dV(\xi) \, d\tau$$

Again, thanks to Gauss' theorem the first term may be transformed into a vanishing surface integral. The underbrace below the second term indicates the for δ/r we may replace $\boldsymbol{\xi}$ by $\boldsymbol{x}$. This is true because the expression δ/r depends on $\boldsymbol{\xi}$ through $r = |\boldsymbol{x} - \boldsymbol{\xi}|$, which shows, that a differentiation w.r.t. $\boldsymbol{x}$ would yield the same result except a minus sign from the inner derivative. Two differentiations therefore yield a factor of unity. After this substitution we see that in the integrand no other term depends on $\boldsymbol{x}$ such that we may take the differentiation outside the integral:

$$\rho'(\boldsymbol{x},t) = \frac{1}{4\pi a_\infty^2} \boldsymbol{\nabla}_x \cdot \boldsymbol{\nabla}_x \cdot \int\limits_{V_S} \frac{\boldsymbol{T}(t - r/a_\infty, \boldsymbol{\xi})}{r}\, dV(\xi) \tag{256}$$

4.1.2 Solution of Lighthill's equation for free field problems in the farfield

Usually we consider the sound field far away from the sources (i.e. for large distances r). For this case it is possible to considerably simplify the solution (256) of Lighthill's wave equation. We rearrange it by taking the gradients inside the integral and two successive applications of the product rule, first

$$\rho'(\boldsymbol{x},t) = \frac{1}{4\pi a_\infty^2} \int\limits_{V_S} \boldsymbol{\nabla}_x \cdot \left[\boldsymbol{\nabla}_x \left(\frac{1}{r}\right) \cdot \boldsymbol{T} + \frac{1}{r} \boldsymbol{\nabla}_x \cdot \boldsymbol{T} \right] dV(\xi)$$

and second:

$$\rho'(\boldsymbol{x},t) = \frac{1}{4\pi a_\infty^2} \int\limits_{V_S} \underbrace{\boldsymbol{\nabla}_x \boldsymbol{\nabla}_x \left(\frac{1}{r}\right)}_{= \frac{3\boldsymbol{e}_r\boldsymbol{e}_r - \boldsymbol{I}}{r^3}} : \boldsymbol{T} + 2 \underbrace{\boldsymbol{\nabla}_x \left(\frac{1}{r}\right)}_{= -\frac{\boldsymbol{e}_r}{r^2}} \cdot \boldsymbol{\nabla}_x \cdot \boldsymbol{T} + \frac{1}{r} \boldsymbol{\nabla}_x \cdot \boldsymbol{\nabla}_x \cdot \boldsymbol{T}\, dV(\xi)$$

Here the underbraces clearly show that the first two terms in the integrand scale with higher powers of the distance r to the source domain V_S than the last term. For $r \to \infty$ they may therefore be ignored:

$$\rho'(\boldsymbol{x},t) \simeq \frac{1}{4\pi a_\infty^2} \int\limits_{V_S} \frac{1}{r} \boldsymbol{\nabla}_x \cdot \boldsymbol{\nabla}_x \cdot \boldsymbol{T}(t - r/a_\infty, \boldsymbol{\xi})\, dV(\xi) \ .$$

Note, that we may replace the gradients (in $\boldsymbol{x}$) on Lighthill's stress tensor $\boldsymbol{T}$ by time derivatives, i.e.

$$\begin{aligned} \boldsymbol{\nabla}_x \cdot \boldsymbol{T}(t - r/a_\infty, \boldsymbol{\xi}) &= -\frac{1}{a_\infty} \boldsymbol{\nabla}_x r \cdot \frac{\partial \boldsymbol{T}}{\partial t} = -\frac{1}{a_\infty}\, \boldsymbol{e}_r \cdot \frac{\partial \boldsymbol{T}}{\partial t} \\ \boldsymbol{\nabla}_x \cdot \boldsymbol{\nabla}_x \cdot \boldsymbol{T}(t - r/a_\infty, \boldsymbol{\xi}) &= \frac{1}{a_\infty^2}\, \boldsymbol{e}_r \boldsymbol{e}_r : \frac{\partial^2 \boldsymbol{T}}{\partial t^2} - \frac{1}{a_\infty} \frac{2}{r} \boldsymbol{e}_r \cdot \frac{\partial \boldsymbol{T}}{\partial t} \end{aligned}$$

The last relation shows again that for large r the second term on the r.h.s. is negligible compared to the first and we may write our farfield solution as

$$\rho'(\boldsymbol{x},t) \simeq \frac{1}{4\pi a_\infty^4} \frac{\partial^2}{\partial t^2} \int\limits_{V_S} \frac{1}{r} \boldsymbol{e}_r \boldsymbol{e}_r : \boldsymbol{T}(t - r/a_\infty, \boldsymbol{\xi})\, dV(\xi) \ .$$

In order to describe large distances between source and observer more explicity we choose a fixed reference point $\boldsymbol{\xi}_0$ somewhere inside the source volume V_S. Its distance vector to the fixed observer is then $\boldsymbol{r}_0 := \boldsymbol{x} - \boldsymbol{\xi}_0$, which may be expressed in terms of $\boldsymbol{r}$ like $\boldsymbol{r}_0 := \boldsymbol{r} + \boldsymbol{\xi} - \boldsymbol{\xi}_0$. Then

$$r_0/r = \sqrt{1 + 2r^{-1}\boldsymbol{e}_r \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}_0) + r^{-2}(\boldsymbol{\xi} - \boldsymbol{\xi}_0)^2} \implies \lim_{r\to\infty} r_0/r = 1 \ ,$$

provided the spatial extent $l_S = \max_{\xi\in V_S} |\boldsymbol{\xi} - \boldsymbol{\xi}_0|$ of the source volume V_S is of course much smaller compared to the large distance r. But this means that also the directions $\boldsymbol{e}_r$ may be simplified because $\boldsymbol{e}_r = \dfrac{\boldsymbol{r}}{r} = \dfrac{\boldsymbol{r}_0}{r_0}\dfrac{r_0}{r} + \dfrac{\boldsymbol{\xi} - \boldsymbol{\xi}_0}{r}$ and thus for large r we obtain $\boldsymbol{e}_r \simeq \boldsymbol{r}_0/r_0 =: \boldsymbol{e}_{r_0}$. This means that we may replace r by $r_0 \neq r_0(\boldsymbol{\xi})$ and $\boldsymbol{e}_r$ by $\boldsymbol{e}_{r_0} \neq \boldsymbol{e}_{r_0}(\boldsymbol{\xi})$ and take these out of the integral. So the acoustic density fluctuation in the farfield finally is

$$\rho'(\boldsymbol{x},t) \simeq \frac{1}{4\pi a_\infty^4 r_0}(\boldsymbol{e}_{r_0}\boldsymbol{e}_{r_0}) \boldsymbol{:} \frac{\partial^2}{\partial t^2} \int\limits_{V_S} \boldsymbol{T}(t - r/a_\infty, \boldsymbol{\xi}) \, dV(\xi) \ . \tag{257}$$

or in the frequency space (Fourier transform) with the wavenumber $k = \omega/a_\infty$:

$$\hat{\rho}(\boldsymbol{x},\omega) \simeq -\frac{k^2}{4\pi a_\infty^2 r_0}(\boldsymbol{e}_{r_0}\boldsymbol{e}_{r_0}) \boldsymbol{:} \int\limits_{V_S} \hat{\boldsymbol{T}}(\omega, \boldsymbol{\xi}) \exp(-ikr) \, dV(\xi) \ .$$

Note that we have not simplified r inside the retarded time $t - r/a_\infty$ at which Lighthill's stress tensor $\boldsymbol{T}$ is to be evaluated in (257). We recall that the retarded time represents phase information (explicitly seen in the term $\exp(-ikr)$ in the frequency domain solution) and given a large enough source volume V_S there may be significant changes of $\boldsymbol{T}$ across this domain.

We try to interpret result (257). The aerodynamically generated acoustic farfield in free space (no scattering objects present) is proportional to the second time derivative of the volume integral over Lighthill's stress tensor, evaluated as integrand at the retarded time $t - r/a_\infty$. As any acoustic field the amplitude decays with the inverse of the distance to the source. The term $(\boldsymbol{e}_r\boldsymbol{e}_r)\boldsymbol{:}$ describes the projection of the various stress components onto the direction of the observer. Alternatively we could have written the far field solution (257) as

$$\rho'(\boldsymbol{x},t) \simeq \frac{1}{4\pi a_\infty^4 r_0} \frac{\partial^2}{\partial t^2} \int\limits_{V_S} T_{r_0 r_0}(t - r/a_\infty, \boldsymbol{\xi}) \, dV(\xi)$$

with $T_{r_0 r_0} := (\boldsymbol{e}_{r_0}\boldsymbol{e}_{r_0})\boldsymbol{:}\boldsymbol{T} = \boldsymbol{e}_{r_0} \cdot (\boldsymbol{T}\boldsymbol{e}_{r_0})$ the observer-oriented component of Lighthill's stress tensor.

4.2 Lilley's equation: Analogy for shear flows

It is a drawback of Lighthill's equation that purely kinematic effects as e.g. refraction at shear-layers are (mis-)interpreted as sources. This is a misinterpretation because these effects appear implicitly as part of Lighthill's stress tensor, although clearly, they are not generating sound. In the attempt to isolate those quantities, which could be identified as true soures of sound, several generialized wave equations have been derived after Lighthill. One important such equation is Lilley's equation. In order that sources could be identified it is common to "shape" the complete conservation equations in such a way that a known wave operator occurs, at least for a special case (concept of an acoustic analogy).

First we identify the sources in (32), (49), (44) with $\dot{m} = 0$, $\boldsymbol{f} = \boldsymbol{\nabla}\cdot\boldsymbol{\tau}$, $\dot{\theta} = \boldsymbol{\tau}:\boldsymbol{\nabla v} - \boldsymbol{\nabla}\cdot\boldsymbol{q}$, respectively. Then (44) yields

$$\underbrace{\frac{1}{\rho a^2}}_{\frac{1}{\gamma p}}\frac{Dp}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{1}{c_p}\frac{1}{\rho}\underbrace{\frac{1}{T}\dot{\theta}}_{=\rho\frac{Ds}{Dt}} = \frac{1}{c_p}\frac{Ds}{Dt}$$

Now we define a new variable $\Pi := \frac{1}{\gamma}\ln\left(\frac{p}{p_\infty}\right)$ and arrive at

$$\frac{D\Pi}{Dt} + \boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{1}{c_p}\frac{Ds}{Dt}$$

On the other hand, from (49) we have:

$$\frac{D\boldsymbol{v}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\tau} \overbrace{=}^{\rho=\frac{\gamma p}{a^2}} -a^2\boldsymbol{\nabla}\Pi + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\tau}$$

In order to eliminate the divergence term in the second last equation, we form the divergence of the above velocity equation:

$$\boldsymbol{\nabla}\cdot\left(\frac{D\boldsymbol{v}}{Dt}\right) = \frac{\partial\boldsymbol{\nabla}\cdot\boldsymbol{v}}{\partial t} + \underbrace{\boldsymbol{\nabla}\cdot(\boldsymbol{v}\cdot\boldsymbol{\nabla v})}_{\frac{\partial}{\partial x_k}\left(v_j\frac{\partial v_k}{\partial x_j}\right) = \underbrace{v_j\frac{\partial}{\partial x_j}\left(\frac{\partial v_k}{\partial x_k}\right)}_{\boldsymbol{v}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})} + \underbrace{\frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_j}}_{\boldsymbol{\nabla v}:{}^t\boldsymbol{\nabla v}}} = -\boldsymbol{\nabla}\cdot\left(a^2\boldsymbol{\nabla}\Pi\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\tau}\right)$$

which yields

$$\frac{D\boldsymbol{\nabla}\cdot\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla v}:{}^t\boldsymbol{\nabla v} - \boldsymbol{\nabla}\cdot\left(a^2\boldsymbol{\nabla}\Pi\right) + \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\tau}\right)$$

Now taking the substantial derivative of the above equation for Π we may finally eliminate the divergence term $\frac{D\boldsymbol{\nabla}\cdot\boldsymbol{v}}{Dt}$ and obtain the "Phillips equation":

$$\frac{D^2\Pi}{Dt^2} - \boldsymbol{\nabla}\cdot\left(a^2\boldsymbol{\nabla}\Pi\right) = \boldsymbol{\nabla v}:{}^t\boldsymbol{\nabla v} - \boldsymbol{\nabla}\cdot\left(\frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\tau}\right) + \frac{1}{c_p}\frac{D^2 s}{Dt^2} \tag{258}$$

This equation closely resembles a wave equation for the quantity Π. However, it was shown that terms associated with the acoustic pressure are still "hidden" in the right hand side term $\boldsymbol{\nabla v}:{}^t\boldsymbol{\nabla v}$. This may be motivated by the same argument which led us from (160) to (199) for the description of sound waves in parallel shear flows. One may still extract terms containing the acoustic variable Π out of $\boldsymbol{\nabla v}:{}^t\boldsymbol{\nabla v}$. Being guided by the derivation of the pressure perturbation equation for parallel shear flows (199), we take the substantial derivative of the Phillips equation (258), in which the essential term is:

$$\begin{aligned}\frac{D}{Dt}(\boldsymbol{\nabla v}:{}^t\boldsymbol{\nabla v}) = \frac{D}{Dt}\Big(\frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_j}\Big) &= \frac{\partial v_i}{\partial x_k}\frac{D}{Dt}\Big(\frac{\partial v_k}{\partial x_i}\Big) + \frac{\partial v_k}{\partial x_i}\frac{D}{Dt}\Big(\frac{\partial v_i}{\partial x_k}\Big)\\ &= 2\frac{\partial v_k}{\partial x_i}\underbrace{\frac{D}{Dt}\Big(\frac{\partial v_i}{\partial x_k}\Big)}_{\frac{\partial}{\partial x_k}\frac{Dv_i}{Dt} - \frac{\partial v_j}{\partial x_k}\frac{\partial v_i}{\partial x_j}}\\ &= 2\,{}^t\boldsymbol{\nabla v}:\boldsymbol{\nabla}\Big(\frac{D\boldsymbol{v}}{Dt}\Big) - 2\,{}^t\boldsymbol{\nabla v}:(\boldsymbol{\nabla v\nabla v})\end{aligned}$$

The substantial derivative of the velocity in this expression may again be eliminated from the velocity relation from above. We may now write down the substantial derivative of the Phillips equation and insert the result to arrive at

$$\begin{aligned}\frac{D}{Dt}\Big[\frac{D^2\Pi}{Dt^2} - \boldsymbol{\nabla}\cdot\Big(a^2\boldsymbol{\nabla}\Pi\Big)\Big] \;&+\; 2\,{}^t\boldsymbol{\nabla v}:\boldsymbol{\nabla}\Big(a^2\boldsymbol{\nabla}\Pi\Big) = -2\,{}^t\boldsymbol{\nabla v}:(\boldsymbol{\nabla v\nabla v}) + \Psi \qquad (259)\\ \Psi \;&:=\; 2\,{}^t\boldsymbol{\nabla v}:\boldsymbol{\nabla}\Big(\frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\tau}\Big) - \frac{D}{Dt}\Big[\boldsymbol{\nabla}\cdot\Big(\frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\tau}\Big) + \frac{1}{c_p}\frac{D^2 s}{Dt^2}\Big]\end{aligned}$$

which is called "Lilley's equation". As Lighthill's wave equation, Lilley's equation is nothing but a re-arrangement of the general conservation equations. As such, there is no simplification contained in it. Clearly, the lengthy term on the left hand side somewhat resembles the wave operator for small pressure perturbations in parallel shear flows (199). This can be verified by evaluating (259) for small perturbations about a mean flow. Before doing so, consider mean Π^0 and perturbation Π' of the acoustic variable $\Pi = \Pi^0 + \epsilon\Pi'$

$$\Pi = \underbrace{\frac{1}{\gamma}\ln\Big(\frac{p^0}{p_\infty}\Big)}_{=:\,\Pi^0} + \epsilon\underbrace{\frac{p'}{\gamma p^0}}_{=:\,\Pi'} + \epsilon^2 ...$$

We are specifically interested in parallel shear flows. In this case the mean flow pressure is constant, so that we can choose $p_\infty = p^0$ and thus by definition $\Pi^0 = 0$. Therefore $\Pi = \epsilon\Pi' = \epsilon p'/\gamma p^0$ to leading order in the Taylor expansion about the small parameter ϵ. Now we can immediately write down the linearized Lilley equation for constant mean flow pressure:

$$\begin{aligned}\frac{D^0}{Dt}\Big[\frac{D^{02}\Pi'}{Dt^2} - \boldsymbol{\nabla}\cdot\Big(a_0^2\boldsymbol{\nabla}\Pi'\Big)\Big] \;&+\; 2\,{}^t\boldsymbol{\nabla v}^0:\boldsymbol{\nabla}\Big(a_0^2\boldsymbol{\nabla}\Pi'\Big) =\\ &-\; 2\,{}^t\boldsymbol{\nabla v}^0:(\boldsymbol{\nabla v}^0\boldsymbol{\nabla v}' + \boldsymbol{\nabla v}'\boldsymbol{\nabla v}^0) - 2\,{}^t\boldsymbol{\nabla v}':(\boldsymbol{\nabla v}^0\boldsymbol{\nabla v}^0) + \Psi'\end{aligned}$$

This equation attains a more simple form if we use the fact that the mean flow is parallel and directed along the x-axis of a coordinate system ,i.e. $\boldsymbol{v}^0 = u^0(h(y,z))\boldsymbol{e}_x$. Inserting this expression into the linearized Lilley equation we obtain

$$\frac{D^0}{Dt}\Big[\frac{D^{02}p'}{Dt^2} - \boldsymbol{\nabla}\cdot\Big(a_0^2\boldsymbol{\nabla}p'\Big)\Big] + 2|\boldsymbol{\nabla}h|\frac{du^0}{dh}a_0^2\frac{\partial^2 p'}{\partial x\partial h} = \gamma p^0\Psi'$$

The left hand side corresponds exactly to the wave operator for small pressure perturbations in parallel shear flows (199). This shows that Lilley's equation represents an acoustic analogy for parallel shear flows.

- Properties of Lilley's analogy equation (259):

  * Equation (259) is an exact consequence of the balance equations of continuum mechanics, i.e. as it stands, no simplification was introduced.

  * As indicated in Lighthill's stress tensor the right hand side of Lilley's equations features three different sources of noise, namely flow turbulence (the term $2\,{}^t\boldsymbol{\nabla v} : (\boldsymbol{\nabla v \nabla v})$), entropy and friction.

  * In contrast to the (standard) wave operator of Lighthill's equation Lilley's wave operator is not self-adjoint. This means, that there will in general exist eigenmodes which get amplified in time and represent hydrodynamic instabilities, which are of purely vortical character. In fact, the linearized Lilley equation is nothing but the inviscid stability equation of hydrodynamic stability theory for parallel laminar flows. From this theory it is e.g. known that mean flow profiles with a certain type of inflexion point are unstable to vortical perturbations, which is always true for free shear layers. For supersonic shear or boundary layers beyond $M \approx 2.2$ there may occur so called "secondary instability modes" or "Mack-Modes" which are associated with sound radiation because these vortical eigenmodes developing in the shear profile move supersonic with respect to the exterior flow and therefore generate Mach waves (just as a supersonically moving wavy wall).

  * The linearized Lilley equation indicates that with the neglect of dissipation and friction terms Ψ' there exist no aeroacoustic sources in free parallel shear flows because for this case the velocity related term on the right hand side drops out identically. In other words, sound generation in parallel shear flows is a fundamentally nonlinear process. This is however not true, if surfaces of objects are present.

- Pros of the analogy concept of Lilley:

  \+ the wave operator describes refraction effects in parallel shear flows; Lilley's source term has therefore a true source character as opposed to Lighthill's source.

- Cons of the analogy concept of Lilley:

  \- there is no general solution known, i.e. a Green's function is unknown. Thus, the solution has to be done numerically (complicated)

  \- As mentioned the wave operator does not only describe acoustic, but vortical degrees of freedom as well, the latter of which may even become unstable. The desired separation between hydrodynamic and acoustic pressure is therefore difficult.

  \- For quiecent medium Lilley's equation does not reduce to the simple wave equation.

4.3 Möhring's wave equation: Analogy for potential flows

If the terms neglected during the derivation of the wave equation for acoustic perturbations in potential flows (237) are identified with the external sources, in particular $\dot{m} = 0$, $\boldsymbol{f} = -\rho\boldsymbol{\omega} \times \boldsymbol{v}$

and $\dot{\theta} = \boldsymbol{\tau} : \boldsymbol{\nabla v} - \boldsymbol{\nabla \cdot q} - \rho T \frac{Ds}{Dt}$, we obtain

$$\frac{\partial}{\partial t}\left[\frac{\rho}{a^2}\frac{DB}{Dt}\right] + \boldsymbol{\nabla\cdot}\left[\frac{\rho \boldsymbol{v}}{a^2}\frac{DB}{Dt}\right] \quad - \quad \boldsymbol{\nabla\cdot}(\rho \boldsymbol{\nabla} B) = Q_M \tag{260}$$

$$Q_M := \underbrace{\boldsymbol{\nabla\cdot}(\rho \boldsymbol{\omega} \times \boldsymbol{v})}_{=: \boldsymbol{L}} + \frac{\partial}{\partial t}\left(\frac{\rho}{a^2} T \frac{Ds}{Dt}\right) + \boldsymbol{\nabla\cdot}\left(\frac{\rho}{a^2}\boldsymbol{v} T \frac{Ds}{Dt}\right)$$

where the stagnation enthalpy B as defined in (225) is the acoustic quantity and $\boldsymbol{L}$ is called "Lamb vector", which plays a fundamental role in the noise generation of flows. The above equation may be slightly simplified if we subtract the mass balance equation (54) pre-multiplied by $\frac{1}{a^2}\frac{DB}{Dt}$:

$$\boxed{\frac{D}{Dt}\left[\frac{1}{a^2}\frac{DB}{Dt}\right] - \frac{1}{\rho}\boldsymbol{\nabla\cdot}(\rho \boldsymbol{\nabla} B) = \frac{1}{\rho} Q_M} \tag{261}$$

This is Möhring's equation.

- Properties of Möhring's equation (261):
  - * Equation (261) is an exact consequence of the balance equations of continuum mechanics, i.e. as it stands, no simplification was introduced.
  - * By comparison of the left hand sides of Möhring's equation and the wave equation for sound propagation in potential flows (240) it is seen, that both are identical. Hence the right hand side may be interpreted as acoustic sources (analogy)
  - * The right hand side of Möhring's acoustic analogy identifies two sources of noise, namely the motion related aeroacoustic sources, represented by the Lamb vector and entropy related sources.
  - * Möhring's source term Q_M shows explicitly that the presence of vorticity is necessary for the generation of aerodynamic sound! Correspondignly aeroacoustic phenomena are often called "vortex sound".

- Pros of the analogy concept of Möhring:
  - + The acoustic wave propagation is described correctly in the potential domain of the flow field, which for external flows is by far the largest part.
  - + Möhring's wave operator is formally self-adjoint, i.e. it does not contain any unstable degrees of freedom, even in arbitrary (e.g. hydrodynamic unstable) flow fields.
  - + Möhring's wave operator reduced to the convective wave equation for uniform flows field
  - + Möhring's wave operator reduced to the classical wave equation for quiecient medium without flow.

- Cons of the analogy concept of Möhring:

- there is no general solution known, i.e. a Green's function is unknown. Thus, the solution has to be done numerically (but numerically "quite well behaved").
- The implementation of physical boundary conditions for B is non trivial, e.g. at surfaces.

4.4 Curle's equation: noise from steady objects in low speed flows

In section 2.6.1 we derived the solution of the wave equation if reflecting and diffracting objects are present in the wave field. In this section we investigate the importance of the presence of objects when the sound sources are aeroacoustic according to Lighthill. We simply use result (128) and apply it for Lighthill's wave equation (252)

$$\rho'(\boldsymbol{x},t) = \int_{-\infty}^{\infty} \Big\{ \int\limits_{V'_\infty} \frac{1}{a_\infty^2} G\, \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{T}\, dV(\xi) + \int\limits_{\partial V_B} \Big(\rho' \frac{\partial G}{\partial n} - G \frac{\partial \rho'}{\partial n} \Big) dS(\xi) \Big\}\, d\tau \qquad (262)$$

Here V'_∞ denotes all of space except the volume of the object(s). As usual the spatial derivatives on the Lighthill tensor $\boldsymbol{T}$ are shifted over to the Green's function G by means of

$$G\, \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{T} = \boldsymbol{\nabla}\boldsymbol{\cdot}(G\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{\cdot}\boldsymbol{\nabla}G) - \boldsymbol{T}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{\nabla}G$$

The first two terms represent a divergence, so that Gauss' theorem can be applied to convert the Volume integral into a surface integral about the bounding surface of V'_∞, shown in figure *46.* This yields

$$\begin{aligned} \rho'(\boldsymbol{x},t) &= \int_{-\infty}^{\infty} \int\limits_{V'_\infty} \frac{1}{a_\infty^2} \boldsymbol{T}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{\nabla}G\, dV(\xi)\, d\tau + \\ &+ \int_{-\infty}^{\infty} \Big\{ \int\limits_{\partial V'_\infty} \frac{1}{a_\infty^2} \Big(G(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{T})\boldsymbol{\cdot}\boldsymbol{n} - (\boldsymbol{T}\boldsymbol{\cdot}\boldsymbol{\nabla}G)\boldsymbol{\cdot}\boldsymbol{n} \Big)\, dS(\xi) + \int\limits_{\partial V_B} \Big(\rho' \frac{\partial G}{\partial n} - G \frac{\partial \rho'}{\partial n} \Big) dS(\xi) \Big\}\, d\tau \end{aligned}$$

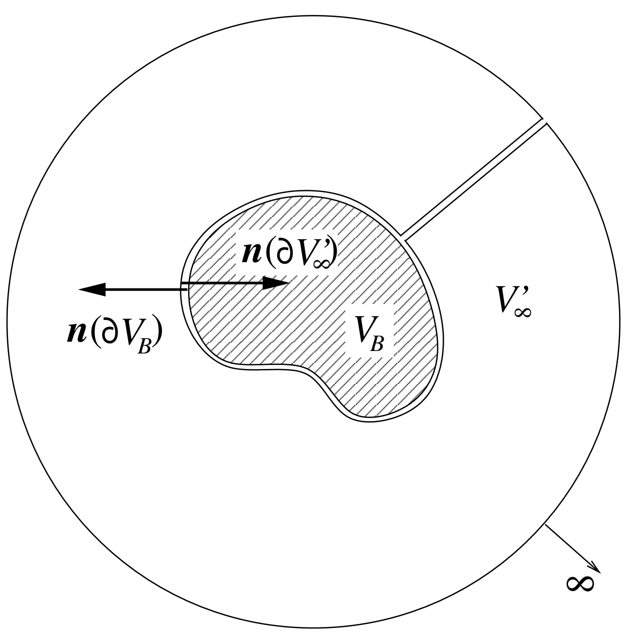

Figure 46: Integration domain

where $\boldsymbol{n}$ denotes a surface unit normal vector. Obviously the integration over the infinite part of V'_∞ is zero, while the integration over the finite part (the surface of the object, where $\boldsymbol{n}(\partial V_B) = -\boldsymbol{n}(\partial V'_\infty)$ may be rearranged:

$$\begin{aligned}\rho'(\boldsymbol{x},t) &= \frac{1}{a_0^2}\int_{-\infty}^{\infty}\int\limits_{V'_\infty}\boldsymbol{T}\boldsymbol{:}\boldsymbol{\nabla\nabla}G\,dV(\xi)\,d\tau + \\ &+ \int_{-\infty}^{\infty}\Big\{\int\limits_{\partial V_B} -G\boldsymbol{\nabla}\boldsymbol{\cdot}\Big(\rho'\boldsymbol{I}+\frac{\boldsymbol{T}}{a_\infty^2}\Big)\boldsymbol{\cdot n} + \boldsymbol{\nabla}G\boldsymbol{\cdot}\Big(\rho'\boldsymbol{I}+\frac{\boldsymbol{T}}{a_\infty^2}\Big)\boldsymbol{\cdot n}\,dS(\xi)\Big\}\,d\tau\end{aligned}$$

now we insert the actual expression for Lighthill's stress tensor $\boldsymbol{T} = \rho\boldsymbol{vv} + (p' - a_\infty^2\rho')\boldsymbol{I} - \boldsymbol{\tau}$ and for G we choose the free field Green's function $G = G_0 = \delta(\tau - t + r/a_\infty)/(4\pi r)$. Then our solution is

$$\begin{aligned}\rho'(\boldsymbol{x},t) &= \frac{1}{4\pi a_\infty^2}\boldsymbol{\nabla}_x\boldsymbol{\cdot}\boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{V'_\infty}\frac{1}{r}\boldsymbol{T}\,dV(\xi) - \\ &- \frac{1}{4\pi a_\infty^2}\int_{-\infty}^{\infty}\int\limits_{\partial V_B}\boldsymbol{\nabla}\Big(\frac{\delta}{r}\Big)\boldsymbol{\cdot}(\rho\boldsymbol{vv}+p'\boldsymbol{I}-\boldsymbol{\tau})\boldsymbol{\cdot n}\,dS(\xi)\,d\tau - \\ &- \frac{1}{4\pi a_\infty^2}\int\limits_{\partial V_B}\frac{1}{r}\underbrace{\int_{-\infty}^{\infty}\delta\,\underbrace{\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{vv}+p'\boldsymbol{I}-\boldsymbol{\tau})}_{(33):\; = -\frac{\partial\rho\boldsymbol{v}}{\partial\tau}}\,d\tau}\,\boldsymbol{\cdot n}\,dS(\xi) \\ &\qquad = -\underbrace{\int_{-\infty}^{\infty}\frac{\partial}{\partial\tau}(\delta\rho\boldsymbol{v})\,d\tau}_{=0} + \int_{-\infty}^{\infty}\rho\boldsymbol{v}\,\underbrace{\frac{\partial\delta(\tau-t+r/a_\infty)}{\partial\tau}}_{= -\frac{\partial\delta}{\partial\tau}}\,d\tau\end{aligned}$$

Denominating the wall normal velocity $v_n := \boldsymbol{n\cdot v}$ with $\boldsymbol{n}$ directed outside the object we arrive at the final, so called "Curle's equation":

$$\begin{aligned}\rho'(\boldsymbol{x},t) &= \frac{1}{4\pi a_\infty^2}\boldsymbol{\nabla}_x\boldsymbol{\cdot}\boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{V'_\infty}\frac{1}{r}\boldsymbol{T}(t-r/a_\infty,\boldsymbol{\xi})\,dV(\xi) - \\ &- \frac{1}{4\pi a_\infty^2}\boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{\partial V_B}\frac{1}{r}\Big(\rho\boldsymbol{v}v_n + (p'\boldsymbol{I}-\boldsymbol{\tau})\boldsymbol{\cdot n}\Big)_{t-r/a_\infty}dS(\xi) + \\ &+ \frac{1}{4\pi a_\infty^2}\frac{\partial}{\partial t}\int\limits_{\partial V_B}\frac{1}{r}\Big(\rho v_n\Big)_{t-r/a_\infty}dS(\xi)\end{aligned} \tag{263}$$

Curle's equation describes the sound field generated by a body immersed in a flow field. According to the three terms three distinct contributions to this sound field are seen. The first term represents nothing but the sound field produced in the free fluid volume, which we know from the solution of Lighthill's equation for the free field. The second term is obviously related to the local forces acting on the surface of the object. In fact, if the body surface is rigid and non-moving, then $v_n = 0$ and the integral represents almost the net aerodynamic force, acting on the fluid. There is however a subtle difference, namely, the terms are to be evaluated at the retarded time and weighted by r^{-1}, which varies along the object. Nevertheless this part of the sound field is called "loading noise". Finally, the third term is related to the displacement of the object. Obviously, if the body is at rest then again $v_n = 0$ and this third contribution of the sound field is zero; it is called "thickness noise".

The acoustic farfield is of main importance. Analogously to the derivation of (257) we consider very large distances to the object and the source: $r \to r_0$, where r_0 is the distance to, say, the geometrical center of gravity of all considered source and object volumes together. We also assume that r_0 is very large compared to the extension of the system of all sources and objects. Then $\boldsymbol{\nabla}_x \to -\dfrac{1}{a_\infty}\boldsymbol{e}_{r_0}\dfrac{\partial}{\partial t}$ with the direction $\boldsymbol{e}_{r_0} := \dfrac{\boldsymbol{r}_0}{r_0}$. Moreover, in the farfield, the density and pressure fluctuations are related according to isentropy via $\rho' = p'/a_\infty^2$, such that Curle's equation yields

$$
\begin{aligned}
p'(\boldsymbol{x},t) \;\simeq\;& \frac{1}{4\pi a_\infty^2 r_0}\,(\boldsymbol{e}_{r_0}\boldsymbol{e}_{r_0})\boldsymbol{:}\,\frac{\partial^2}{\partial t^2}\int\limits_{V'_\infty}[\boldsymbol{T}]_{t-r/a_\infty}\,dV(\xi)\,+ \\
&+\;\frac{1}{4\pi a_\infty r_0}\,\boldsymbol{e}_{r_0}\boldsymbol{\cdot}\,\frac{\partial}{\partial t}\int\limits_{\partial V_B}\left[\rho\boldsymbol{v}v_n + (p'\boldsymbol{I}-\boldsymbol{\tau})\boldsymbol{n}\right]_{t-r/a_\infty}dS(\xi)\,+ \\
&+\;\frac{1}{4\pi r_0}\,\frac{\partial}{\partial t}\int\limits_{\partial V_B}\left[\rho v_n\right]_{t-r/a_\infty}dS(\xi)
\end{aligned}
\qquad (264)
$$

4.5 Ffowcs-Williams and Hawkings equation: noise from objects in motion

The influence of motion on the sound radiation from point sources was investigated in section 3.3.1. As a result we learned, that even stationary sources (mass, heat or momentum) generate sound whenever in accelerated (or supersonic) motion. This points to a potential sound generation mechanism of objects in accelerated motion, because their action on the surrounding fluid may be described by an integration over point sources (see potential flow theory). For instance, the displacement of a body in a flow may be described by a spatial distribution of mass sources. As long as the net mass flow of all contributing sources is zero, a body with a closed shape may be represented. We want to generalize this for the case of moving objects with a finite extension. We consider a situation as sketched in figure *17*, where a (closed) control surface ∂V_H surrounds the object completely. It may or may not surround the aerodynamically induced sound (volume) sources. The control surface is moving with $v_H(\boldsymbol{x},t)$.

Just as for the derivation of the Kirchhoff equation in section 2.6.3 we want to describe the sound field outside of the control surface ∂V_H by information given only on this very surface. In order to define all variables in all of space (including the inside of the body V_B) we introduce generalized variables using $H(f)$ as a pre-factor to all relevant quantities. For the density perturbation $\rho' := \rho - \rho_\infty$ about its free field value ρ_∞ we use now instead $H(f)\rho'$. Next we try to derive an equation for this new variable by multiplying the mass balance (32) by $H(f)$ and applying the product rule (no mass sources $\dot{m} = 0$):

$$H\Big(\frac{\partial \rho'}{\partial t} + \boldsymbol{\nabla}(\rho \boldsymbol{v})\Big) = \frac{\partial H\rho'}{\partial t} - \rho'\frac{\partial H}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(H\rho\boldsymbol{v}) - \rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}H = 0$$

This leads to new mass balance equation with a non-zero right hand side:

$$\frac{\partial(H\rho')}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(H\rho\boldsymbol{v}) = \rho'\frac{\partial H}{\partial t} + \rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}H$$

in which we may still replace $\frac{\partial H}{\partial t} = \frac{dH}{df}\frac{\partial f}{\partial t} = \delta(f)\frac{\partial f}{\partial t}$. Moreover from relation (248) we have $\frac{\partial f}{\partial t} = -\boldsymbol{v}_H\boldsymbol{\cdot}\boldsymbol{\nabla}f$, which finally yields

$$\frac{\partial(H\rho')}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(H\rho\boldsymbol{v}) = [\rho(\boldsymbol{v}-\boldsymbol{v}_H) + \rho_\infty\boldsymbol{v}_H]\boldsymbol{\cdot}\boldsymbol{\nabla}f\delta(f)$$

The same procedure may be applied to the momentum balance (33), in which p may be replaced by $p' = p - p_\infty$ to give

$$\frac{\partial(H\rho\boldsymbol{v})}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(H\rho\boldsymbol{v}\boldsymbol{v} - H\boldsymbol{\tau}) + \boldsymbol{\nabla}(Hp') = [\rho\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}_H) - \boldsymbol{\tau} + p'\boldsymbol{I}]\boldsymbol{\nabla}f\delta(f)$$

Just as in the derivation of Lighthill's wave equation we may now eliminate the term $H\rho\boldsymbol{v}$ between the last two relations. We take the time derivative of the former and subtract the divergence of the latter to obtain

$$\begin{aligned}\frac{\partial^2}{\partial t^2}(H\rho') - a_\infty^2\Delta(H\rho') \quad &= \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}(H\boldsymbol{T}) - \\ &- \quad \boldsymbol{\nabla}\boldsymbol{\cdot}\Big(\Big[\rho\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}_H) - \boldsymbol{\tau} + p'\boldsymbol{I}\Big]\boldsymbol{\nabla}f\delta(f)\Big) \\ &+ \quad \frac{\partial}{\partial t}\Big(\Big[\rho(\boldsymbol{v}-\boldsymbol{v}_H) + \rho_\infty\boldsymbol{v}_H\Big]\boldsymbol{\cdot}\boldsymbol{\nabla}f\delta(f)\Big)\end{aligned} \tag{265}$$

This is the Ffowcs-Williams & Hawkings (FW-H) wave equation (1969). Its first line resembles (yet due to $H(f)$ does not exactly equal) Lighthill's equation (252), whenever aerodynamic sources are outside the control volume V_H. However, there appear two extra terms on the right hand (i.e. source) side. Both are non-zero only on the surface of the control volume ∂V_H.

We solve this wave equation using the free field Green's function $G_0 = \delta(g)/(4\pi r)$ with $g = \tau - t + r/a_\infty$ and immediately obtain

$$\begin{aligned}4\pi a_\infty^2 H\rho'(\boldsymbol{x},t) \quad &= \quad \boldsymbol{\nabla}_x\boldsymbol{\cdot}\boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{-\infty}^{\infty}\int\limits_{V_\infty}\frac{H\boldsymbol{T}}{r}\delta(g)\,dV(\xi)\,d\tau - \\ &- \quad \boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{-\infty}^{\infty}\int\limits_{V_\infty}\Big[\rho\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}_H) - \boldsymbol{\tau} + p'\boldsymbol{I}\Big]\boldsymbol{n}\,|\boldsymbol{\nabla}_\xi f|\,\delta(f)\frac{\delta(g)}{r}\,dV(\xi)\,d\tau + \\ &+ \quad \frac{\partial}{\partial t}\int\limits_{-\infty}^{\infty}\int\limits_{V_\infty}\Big[\rho(\boldsymbol{v}-\boldsymbol{v}_H) + \rho_\infty\boldsymbol{v}_H\Big]\boldsymbol{\cdot}\boldsymbol{n}\,|\boldsymbol{\nabla}_\xi f|\,\delta(f)\frac{\delta(g)}{r}\,dV(\xi)\,d\tau\end{aligned}$$

where we replaced $\boldsymbol{\nabla}_\xi f = \boldsymbol{n}|\boldsymbol{\nabla}_\xi f|$, $\boldsymbol{n}$ representing the unit normal pointing outward to V_H.

Note that the integrands of the second and third integral are different from zero when $f(\boldsymbol{\xi}, \tau) = 0$ only. This in turn means that in order to satisfy $f = 0$ the positions $\boldsymbol{\xi}$ have to be chosen dependent on time τ. Therefore it is resonable to consider $\boldsymbol{\xi} = \boldsymbol{\xi}(\tau)$ with $\boldsymbol{\xi}(\tau) = \int_{\tau_0}^{\tau} \boldsymbol{v}_H d\tau^* + \boldsymbol{\xi}(\tau_0)$, such that $\frac{\partial}{\partial \tau}\boldsymbol{\xi} = \boldsymbol{v}_H$ and $\boldsymbol{\xi}(\tau_0)$ describing the surface $f(\boldsymbol{\xi}, \tau_0) = 0$ at some intermediate instant τ_0.

The position vector $\boldsymbol{\xi}$ may be expressed in a fixed (time independent) observer co-ordinate base system $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, implying that the coordinate values depend on time, which is rather inconvenient for the execution of the integrals. Instead a co-ordinate system $\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3$ may be used, which is attached to the moving control surface, i.e. a system, in which the description of the control surface ∂V_H appears temporally constant.The position vectors $\boldsymbol{\xi}$ may now be expressed in these two systems

$$\boldsymbol{\xi} = \xi_1(\tau)\boldsymbol{e}_1^\xi + \xi_2(\tau)\boldsymbol{e}_2^\xi + \xi_3(\tau)\boldsymbol{e}_3^\xi = \eta_1\boldsymbol{g}_\eta^1(\tau) + \eta_2\boldsymbol{g}_\eta^2(\tau) + \eta_3\boldsymbol{g}_\eta^3(\tau) + \boldsymbol{\eta}_0(\tau) \tag{266}$$

Here $\boldsymbol{e}_k^\xi$ denotes the k'th unit basis vector of the non-moving cartesian system ξ, while $\boldsymbol{g}_\eta^k(\tau)$ represents the (generally non-unit) basis vector of the moving co-ordinate system. The position vector $\boldsymbol{\eta}_0$ has the meaning of the (moving) origin of the η-system (e.g. geometric center of gravity of V). Note, that "fixing the co-ordinate system to the surface" means, that in (266) the co-ordinates do not depend on time $\eta_k \neq \eta_k(\tau)$; any rigid body motion and/or deformation is accounted for by the time-varying basis $\boldsymbol{g}_\eta^k(\tau)$. If we want to transform the volume integral in the solution of the FW-H wave equation from $\boldsymbol{e}_k^\xi$ to $\boldsymbol{g}_\eta^k$ we need to account for the "functional determinant" or "Jacobian"

$$J := \det\left(\frac{\partial \xi_i}{\partial \eta_j}\right) = \frac{dV(\xi)}{dV(\eta)}$$

which represents the ratio of the differential volume elements in the two systems. We may use (266) to express ξ_i in terms of η_k by forming the dot product with $\boldsymbol{e}_i^\xi$; then we have

$$\xi_i(\tau) = \eta_k(\boldsymbol{g}_\eta^k(\tau)\cdot\boldsymbol{e}_i^\xi)$$

The differentiation with respect to η_j is now easily done and yields

$$\frac{\partial \xi_i}{\partial \eta_j} = \boldsymbol{g}_\eta^j(\tau)\cdot\boldsymbol{e}_i^\xi \tag{267}$$

We may now solve the time integral in the above solution of the FW-H wave equation. It is important to note that the argument $g = \tau - t + r/a_\infty$ depends on the source time τ additionally implicit through $r = |\boldsymbol{x} - \boldsymbol{\xi}(\tau)|$. Rule (114) brings the convection factor

$$\left|\frac{dg}{d\tau}\right| = \left|1 + \frac{1}{a_\infty}\underbrace{\frac{\partial \boldsymbol{\xi}}{\partial \tau}}_{\boldsymbol{v}_H}\cdot\underbrace{\boldsymbol{\nabla}_\xi r}_{-\boldsymbol{e}_r}\right| = |1 - M_r|$$

into play. $M_r = \boldsymbol{v}_H\cdot\boldsymbol{e}_r/a_\infty$ denotes the Mach number component of the volume (or surface element) in the direction of the observer.

When solving the volume integral it is necessary to respect the delta function $\delta(f)$, which by rule (117) brings a factor $|\boldsymbol{\nabla}_\eta f|$ in the denominator into play, such that finally we have

$$\begin{aligned} 4\pi a_\infty^2 H\rho'(\boldsymbol{x},t) &= \nabla_x \cdot \nabla_x \cdot \int\limits_{V_H^+} \frac{\boldsymbol{T}}{r|1-M_r|}\, J\, dV(\eta) - \\ &- \nabla_x \cdot \int\limits_{\partial V_H} \frac{[\rho \boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}_H) - \boldsymbol{\tau} + p'\boldsymbol{I}]\boldsymbol{n}}{r|1-M_r|}\, K\, dS(\eta) + \\ &+ \frac{\partial}{\partial t} \int\limits_{\partial V_H} \frac{[\rho(\boldsymbol{v}-\boldsymbol{v}_H) + \rho_\infty \boldsymbol{v}_H]\cdot \boldsymbol{n}}{r|1-M_r|}\, K\, dS(\eta) \end{aligned} \quad (268)$$

with the abbreviations $K := J\,|\nabla_\xi f|/|\nabla_\eta f|$ and $V_H^+ = V_\infty - V_H$ the volume outside of ∂V_H. Note, that all integrands are to be evaluated at the retarded time $t - r/a_\infty$. This is the famous Ffowcs-Williams & Hawkings equation in general form for an integration surface ∂V_H in arbitrary motion (also called "porous (or permeable) FW-H equation"). Note that if all the aerodynamic sources (volume integral) are contained inside ∂V_H, then the first term on the right hand side drops out and the sound field is determined completely by the two surface integrals. Due to the motion of the surface a convective amplification is seen through the convection factor $|1 - M_r|$.

We still need to interpret the term $K = J\,|\nabla_\xi f|/|\nabla_\eta f|$ in (268). We represent the gradient of f in the ξ and η systems respectively using the introduced coordinate transform:

$$\nabla_\eta f = \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\eta}}\, \nabla_\xi f, \quad \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\eta}} := \frac{\partial \xi_i}{\partial \eta_k}(\boldsymbol{g}_\eta^k\, \boldsymbol{e}_i^\xi)$$

Now we take the magnitude of this relation and divide by $|\nabla_\xi f|$ to obtain

$$\frac{|\nabla_\eta f|}{|\nabla_\xi f|} = \left|\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\eta}}\boldsymbol{n}\right| =: \frac{d\xi_n}{d\eta_n}$$

which uses the transformation from the $\eta-$system to the $\xi-$system of the unit normal vector $\boldsymbol{n}$ to the level surfaces of f to show that $|\nabla_\xi f|/|\nabla_\eta f|$ has the meaning of the stretch factor for the magnitude of the normal vector in the η system in relation to the ξ system. On the other hand we may represent the volume elements like $dV(\xi) = d\xi_n dS(\xi)$ and $dV(\eta) = d\eta_n dS(\eta)$. Therefore the term $K = |\nabla_\xi f|/|\nabla_\eta f| J = dS(\xi)/dS(\eta)$ represents the area ratio of the surface elements in the non-moving and moving coordinates; this expression is also called "surface dilatation" [6].

One may draw back the integration surface in the permeable FW-H equation (268) onto the surface of the object: $\partial V_H = \partial V_B$. Here the local surface velocity has to assume the local surface velocity of the body, which in turn equals the local fluid velocity $\boldsymbol{v}_H = \boldsymbol{v}$ due to the no slip condition. Moreover, if the object is moving and deforming while conserving its volume (incompressible motion), then $J = 1$, but generally $|\nabla_\xi f|/|\nabla_\eta f| \neq 1$, because the local curvature of the body surface is changing. This is e.g. the case for an elastically deforming and rotating slender helicopter blade.

[6] if ∂V_H was a sphere with time varying radius $R(\tau)$, then $dS(\xi) = du_\xi\, dv_\xi$ with the two (mutually orthogonal) line elements du_ξ, dv_ξ tangent to ∂V_H, while $dS(\eta) = du_\eta dv_\eta$. Here, $(du_\xi, dv_\xi) = \nu(du_\eta, dv_\eta)$ with $\nu = R/R_0$, in which $R_0 \neq R_0(\tau)$ corresponds to the actual radius at some time τ_0. In this case $K = (R/R_0)^2$ and $J = (R/R_0)^3$

If the object is in a rigid body motion, then this motion $\boldsymbol{v}_H$ is a superposition of a translation $\boldsymbol{v}_T$ and a rotation, which for any moment in time can be represented by a pure rotation about an instantaneous pole by a rotation matrix $\boldsymbol{\Omega}$. Note that the rotation matrix $\boldsymbol{\Omega}$ may be composed out of three subsequent rotations about the three co-ordinate axes $\boldsymbol{e}_1^\xi, \boldsymbol{e}_2^\xi, \boldsymbol{e}_3^\xi$,, i.e. $\boldsymbol{\Omega} = \boldsymbol{\Omega}^1\boldsymbol{\Omega}^2\boldsymbol{\Omega}^3$. The determinant of each of these single rotation matrices is equal to unity and so is $\det \boldsymbol{\Omega}$. Therefore $J = \det \boldsymbol{\Omega} = 1$ in this case. Of course, for a rigid body motion the local surface element cannot change either, which means that $K = 1$. The FW-H equation for a rigid body motion (e.g. propeller blades) and $V_H = V_B$ is therefore

$$\begin{aligned} 4\pi a_\infty^2 H\rho'(\boldsymbol{x},t) &= \boldsymbol{\nabla}_x\boldsymbol{\cdot}\boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{V_B^+} \frac{\boldsymbol{T}}{r|1-M_r|}\,dV(\eta) - \\ &- \boldsymbol{\nabla}_x\boldsymbol{\cdot}\int\limits_{\partial V_B} \frac{(-\boldsymbol{\tau}+p'\boldsymbol{I})\boldsymbol{n}}{r|1-M_r|}\,dS(\eta) + \\ &+ \frac{\partial}{\partial t}\int\limits_{\partial V_B} \frac{\rho_\infty v_n}{r|1-M_r|}\,dS(\eta) \end{aligned} \tag{269}$$

As in all acoustic integrals, the integrand is to be evaluated at the retarded time $t - r/a_\infty$. The first term is usually called ”quadrupole noise” (contribution of the unsteady flow volume), the second term is called ”loading noise” (note $p' = p - p_\infty$ typically is not a small deviation from p!), while the third is called ”thickness noise”. Similarly to Curle’s equation, the second term is roughly related to the aerodynamic load on the object, while the third term is related to the (unsteady) volume displacement of the body in the medium. Note, that this third term only depends on the geometry and the kinematics of the body motion. For non-moving objects the thickness noise vanishes.

In the farfield the solution (269) becomes:

$$\begin{aligned} a_\infty^2\rho'(\boldsymbol{x},t) = p'(\boldsymbol{x},t) &\simeq \frac{1}{4\pi a_\infty^2 r_0}\,(\boldsymbol{e}_{r_0}\boldsymbol{e}_{r_0})\boldsymbol{:}\frac{\partial^2}{\partial t^2}\int\limits_{V_B^+} \frac{\boldsymbol{T}}{|1-M_r|}\,dV(\eta) + \\ &+ \frac{1}{4\pi a_\infty r_0}\,\boldsymbol{e}_{r_0}\boldsymbol{\cdot}\frac{\partial}{\partial t}\int\limits_{\partial V_B} \frac{(p'\boldsymbol{I}-\boldsymbol{\tau})\boldsymbol{n}}{|1-M_r|}\,dS(\eta) + \\ &+ \frac{1}{4\pi r_0}\frac{\partial}{\partial t}\int\limits_{\partial V_B} \frac{\rho_\infty v_n}{|1-M_r|}\,dS(\eta) \end{aligned} \tag{270}$$

where $v_n = \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{n}$ denotes the velocity component normal on the surface element due to the motion of the aerodynamic surface, $\boldsymbol{e}_{r_0}$ is the unit vector pointing from the center of the source to the observer and $\boldsymbol{I}$ is the unit matrix. It is reminded that the integrands are to be evaluated at the retarded time $\tau = t - r/a_\infty$ (note, here it is important to use the true distance r instead of r_0 because even small differences in the distance across the source domain may introduce significant phase shifts in the signals produced by the various source locations).

It is easier to physically interpret the contributions of the FW-H equation if one takes the derivatives in (268) inside the integral. If the surface motion, again, is a rigid body motion, then one obtains

$$a_\infty^2 \rho'(\boldsymbol{x},t) = \frac{1}{4\pi a_\infty^2} \int\limits_{V_H^+} \frac{\ddot{T}_{rr}}{r|1-M_r|^3} + \frac{3\dot{T}_{rr}\dot{M}_r + T_{rr}\ddot{M}_r}{r|1-M_r|^4} + \frac{3T_{rr}\dot{M}_r^2}{r|1-M_r|^5} + T_{nf}\, dV(\eta) +$$
$$+ \frac{1}{4\pi a_\infty} \int\limits_{\partial V_H} \frac{\dot{f}_r}{r|1-M_r|^2} + \frac{f_r \dot{M}_r}{r|1-M_r|^3} + f_{nf}\, dS(\eta) +$$
$$+ \frac{1}{4\pi} \int\limits_{\partial V_H} \frac{\dot{m}_n}{r|1-M_r|^2} + \frac{m_n \dot{M}_r}{r|1-M_r|^3} + m_{nf}\, dS(\eta)$$

in which T_{nf}, f_{nf}, m_{nf} are near field terms $\sim r^{-n}$ with $n > 1$, listed in appendix E. Moreover the abbreviations

$$\begin{aligned} f_r &:= \boldsymbol{e}_r \cdot ([\rho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{v}_H) - \boldsymbol{\tau} + p'\boldsymbol{I}]\boldsymbol{n}) \\ m_n &:= [\rho(\boldsymbol{v} - \boldsymbol{v}_H) + \rho_\infty \boldsymbol{v}_H] \cdot \boldsymbol{n} \\ T_{rr} &:= \boldsymbol{e}_r \cdot (\boldsymbol{T}\boldsymbol{e}_r) \end{aligned}$$

were used. The subscripts r at either vectorial or tensorial quantities always denote the projection(s) onto the direction $\boldsymbol{e}_r$ and the dot denotes the derivative w.r.t. the emission time $\tau = t - r/a_\infty$.

For the derviation of the above formulation of the FW-H, one uses the fact that in the co-moving system η the integration surface is not dependent on either $\boldsymbol{x}$ or t (remember $dS(\eta) \neq dS(\tau)$ and hence $\partial V_H(\eta)$). Expressions occur, where the gradient of r is to be taken. When doing so, one has to respect the fact that $r = |\boldsymbol{x} - \boldsymbol{\xi}(\tau)|$, while τ itself again depends on $\boldsymbol{x}$ through $\tau = t - r/a_\infty$! Therefore

$$\boldsymbol{\nabla}_x r = \boldsymbol{\nabla}_x \sqrt{(\boldsymbol{x}-\boldsymbol{\xi})^2} = \frac{[\boldsymbol{\nabla}_x(\boldsymbol{x}-\boldsymbol{\xi})](\boldsymbol{x}-\boldsymbol{\xi})}{\sqrt{(\boldsymbol{x}-\boldsymbol{\xi})^2}} = \frac{[\boldsymbol{\nabla}_x \boldsymbol{x} - \boldsymbol{\nabla}_x \boldsymbol{\xi}](\boldsymbol{x}-\boldsymbol{\xi})}{r} = \frac{\boldsymbol{I}\boldsymbol{r} - \frac{d\boldsymbol{\xi}}{d\tau} \cdot \boldsymbol{r} \boldsymbol{\nabla}_x \tau}{r} =$$
$$= \boldsymbol{I}\boldsymbol{e}_r - \boldsymbol{v}_H \cdot \boldsymbol{e}_r (-\boldsymbol{\nabla}_x r / a_\infty) = \boldsymbol{e}_r + M_r \boldsymbol{\nabla}_x r \Rightarrow \boldsymbol{\nabla}_x r = \frac{\boldsymbol{e}_r}{1 - M_r}$$

In a similar way, for the time derivative w.r.t. t, one has to take into account that the integrand in the (second) integral is rather dependent on τ. Therefore one may use the relation

$$\frac{\partial}{\partial t} = \frac{d\tau}{dt}\frac{\partial}{\partial \tau}, \quad \tau = t - r(\tau)/a_\infty \Rightarrow \frac{d\tau}{dt} = 1 - \frac{1}{a_\infty} \underbrace{\boldsymbol{\nabla}_\xi r}_{-\boldsymbol{e}_r} \frac{d\boldsymbol{\xi}}{d\tau}\frac{d\tau}{dt} \Rightarrow \frac{\partial}{\partial t} = \frac{1}{1 - M_r}\frac{\partial}{\partial \tau}$$

i.e. replacing the derivative w.r.t. t with the one to τ yields another time the convection factor.

It is interesting to see that each source element dS of the first surface integral in this formulation of the FW-H equation exactly acts as a point force with the strength $\boldsymbol{f}_p = [\rho \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{v}_H) - \boldsymbol{\tau} + p'\boldsymbol{I}] d\boldsymbol{S}$ (compare to (242)). Likewise, each source element dS of the second surface integral exactly acts as a point mass source with the strength $m_p = [\rho(\boldsymbol{v} - \boldsymbol{v}_H) + \rho_\infty \boldsymbol{v}_H] \cdot d\boldsymbol{S}$, compare to (243).

5 Technical application

5.1 Jet noise

The noise levels of civil jet aircraft introduced in the 1950'ies were so extreme that this problem triggered the research on noise generated aerodynamically. This was jet noise and in the following we use (257) to estimate the sound intensity as a function of the jet flow speed $\rho'(\boldsymbol{x}, t) \simeq \dfrac{1}{4\pi a_\infty^4 r_0}(\boldsymbol{e}_{r_0}\boldsymbol{e}_{r_0})\boldsymbol{:}\ \dfrac{\partial^2}{\partial t^2}\int_{V_S} \boldsymbol{T}\ dV$. Here we are not interested in quantitative prediction but the way in which the noise scales with the geometrical and operational parameters of the jet flow.

We assume to have a cold flow, such that entropy related noise may be neglected. With the usual simplification of a negligible contribution of the viscous stress to the sources Lighthill's stress tensor is $\boldsymbol{T} = \rho\boldsymbol{v}\boldsymbol{v}$. We consider the density $\rho \simeq \rho_\infty$ and the velocity $\boldsymbol{v}$ scales like the mean flow jet speed u_s. This leads us to estimate the Lighthill stress tensor as $T_{ij} \sim \rho_\infty u_s^2$. Next we need to determine, how the temporal change, expressed as $\dfrac{\partial^2}{\partial t^2}$, scales with the operating parameters of the jet. For this purpose and in a most simplifying way consider turbulence as a random sequence of eddies, i.e. vortices. When focussing on a source element in the turbulent flow, the characteristic frequency $1/t_c$ generated by an eddy passing through the position of the source element is proportional to the mean convection speed u_c of the eddy and is inversely proportional to its characteristic dimension l, i.e. $1/t_c \sim u_c/l$. Now u_c clearly scales with u_s and the eddy size l scales like the characteristic dimension of the jet, which is its nozzle diameter D (the larger the nozzle, the larger the eddies). We therefore find the characteristic scaling $\frac{\partial^2}{\partial t^2} \sim (u_s/D)^2$. Finally, the volume to be integrated $\int ...dV$ scales like D^3. Now all components in the farfield expression for the acoustic density fluctuation have been estimated and we obtain

$$\rho' \sim \frac{1}{a_\infty^4}\frac{1}{r_0}\left(\frac{u_s}{D}\right)^2 \rho_\infty u_s^2 D^3$$

In the farfield the intensity $I = \overline{p'v_r'} \simeq \overline{p'^2}/\rho_\infty a_\infty$ and due to isentropy $p' = a_\infty^2\rho'$ which yields $I \simeq \overline{\rho'^2} a_\infty^3/\rho_\infty$ and thus

$$I \sim \frac{1}{r_0^2}\frac{\rho_\infty}{a_\infty^5} D^2 u_s^8 \tag{271}$$

- This famous result derived by Lighthill says that the sound intensity scales with the 8'th power of the jet flow velocity.
- For high supersonic Mach numbers of the flow $u_s/a_\infty > 2$ the 8'th power law transitions into u_s^3.
- the u_s^8-law is valid for free turbulence in general
- if the sound generation of enptropy fluctuations (neglected here) is significant this lowers the exponent

5.2 Noise from compact objects

Fixed and rigid objects generate excess noise due to the interaction of the body surface with unsteady flow fluctuations (e.g. turbulence). The far field solution of Curle's or Ffowcs-Williams and Hawkings equation (264) or (270) respectively, is

$$p' \simeq \frac{1}{4\pi a_\infty^2 r_0} (\boldsymbol{e}_{r_0}\boldsymbol{e}_{r_0})\boldsymbol{:} \frac{\partial^2}{\partial t^2} \int_{V'_\infty} \boldsymbol{T}\, dV + \frac{1}{4\pi a_\infty r_0} \boldsymbol{e}_{r_0}\boldsymbol{\cdot} \frac{\partial}{\partial t} \int_{\partial V_B} (p\boldsymbol{I} - \boldsymbol{\tau})\boldsymbol{\cdot n}\, dS.$$

The first term yields contributions as in free turbulence (see section 5.1). The excess noise due to the presence of the object is described by the surface integral term, which we will concentrate on for the moment. We are interested only in parts of the sound signal below a characteristic frequency $f_c = 1/t_c$ which corresponds to the characteristic time which a sound signal needs to travel across the largest extension D of the object: $f_c = a_\infty/D$. For frequencies $f << f_c$ the retarded time differences over the body may be neglected and the object is called "compact body". Then the surface integral of Curle's equation represents the net aero force on the body $\boldsymbol{F} \simeq \int_{\partial V_B} (p\boldsymbol{I} - \boldsymbol{\tau})\boldsymbol{n}\, dS$. The compactness of a source or body is characterized by the dimensionless parameter

$$He := \frac{D}{\lambda}$$

where $\lambda = a_\infty/f$ is the wave length. He is called "Helmholtz number". For $He << 1$ the body is compact. For compact bodies the surface related sound source in Curle's / Ffowcs-Williams and Hawkings equation reduces to $\frac{1}{a_\infty r_0} \boldsymbol{e}_{r_0}\boldsymbol{\cdot} \frac{d\boldsymbol{F}}{dt} = \frac{1}{a_\infty r_0} \frac{dF_x}{dt}$, where F_x denotes the component of the aero force in the direction of the observer. This means that the sound field is directly proportional to the time change of the net aero force on the body.

Let us again estimate the magnitude of the aerodynamic force. It is mainly generated by the surface pressure p , which scales like $p \sim \rho_\infty U_\infty^2$. For a body of characteristic dimension D the force $|\boldsymbol{F}|$ scales like $|\boldsymbol{F}| \sim \rho_\infty U_\infty^2 D^2$. The time change of the force occurs during the characteristic period t_c it takes to convect a flow disturbance (e.g. turbulence element) past the body, i.e. $t_c \sim D/U_\infty$. From this we have $\frac{\partial}{\partial t} \sim 1/t_c = U_\infty/D$. Finally, the sound intensity of a compact body scales like

$$I \sim \frac{1}{r_0^2} \frac{\rho_\infty}{a_\infty^3} D^2 U_\infty^6 \tag{272}$$

- the sound intensity of compact bodies scales with the sixth power of the flow speed
- for small (subsonic) flow Mach number the sound signals originating from the surface dominate the overall sound field. The quadrupole terms are negligible (factor $\sim M^2$ smaller).
- the sound field of a compact body has a directivity like a dipole. The dipole axis is aligned with the (unsteady) aero force on the body.

Note that in the aeolian tone problem of a cylinder in cross flow (section 2.6.4.3) we observed exactly this sixth power dependence of the sound power from the flow speed for relatively short, i.e. compact cylinders.

5.3 Noise from non-compact objects

For non-compact objects the exponent of the flow speed scaling law changes. Ffowcs-Williams and Hall (1970) showed for instance, that the sound intensity generated at the trailing edge of a flat plate of very large extent immersed in a turbulent flow scales like U_∞^5 instead of U_∞^6 as for a compact body.

- The presence of the trailing edge in the turbulent flow field increases the efficiency, by which turbulent fluctuations are converted into sound like $I_{trailingedge}/I_{freeturbulence} \sim u_\infty^5/U_\infty^8 = M^{-3}$.
- As a general rule sound is generated mainly at positions where turbulent flow experiences an abrupt change in the boundary conditions (sudden absence of the hard wall condition for a turbulence element passing across an edge).
- slanting a trailing edge has a sound reducing effect because the change in boundary condition seen by the convecting turbulence is more smooth (serrations). The edge noise mechanism is discussed in detail in the lecture ”Methods of Aeroacoustics”.

6 Further reading

Apart from the cited literature, we like to recommend certain books or scripts, we consider especially helpful when studying aeroacoustics.

- The script ”An Introduction to Acoustics” by S.W. Rienstra and A. Hirschberg, Eindhoven University of Technology, P.O. Box 513, 5600 MB EInhoven, The Netherlands is a superb introduction not only to classical acoustics, but to the basics of aeroacoustics. The authors develop everything from basic principles and all mathematical tools used are explained, understandable from an average engineering mathematics background. Many practical examples serve to deepen the understanding of the presented the material.
- The textbook ”Sound and Sources of Sound” by A.P. Dowling and J.E. Ffowcs Williams (Ellis Horwood Limited, distributors John Wiley& Sons, 1983) is very well readable and gives a broad and sufficiently deep overview on what the title says.
- Reference [1] by Crighton et al. is a set of well selected lectures on ”Modern Methods in Analytical Acoustics”. The book covers a wide range of topics, related to acoustics. The topics are elaborated from basics to pretty advanced techniques. Everything, especially the advanced mathematical tools are exlained and made understandable for readers with a typical engineering knowledge of mathematics. The text gives a very good overview about the pertinent methods in use, and so it also represents a good reference book for applied mathematical methods.
- M.E. Goldstein’s classical textbook ”Aeroacoustics” (McGraw-Hill 1976) is probably the most known book on aeroacoustics. It provides a broad range of aeroacoustics topics and methods. The mathematics used go beyond a standard engineering knowledge and are sometimes kept rather brief.
- The classical textbook ”Theoretical Acoustics” by P.M. Morse and K.U. Ingard (McGraw-Hill 1968) is very elaborate. Very many aspects of acoustics including mathematical methods are presented, and aeroacoustics appears as one of the several sub-topics. The book is very mathematical and requires respective skills of the reader.
- The book ”Wind turbine noise” by S. Wagner, R. Bareiss, G. Guidati (Springer Verlag 1996) provides a very well readable introduction to aeroacoustics. Only the second half of the book is specialized on wind turbines.

References

[1] Crighton, D.G.; Dowling, A.P.; Ffowcs Williams, J.E.; Heckl, M.; Leppington, F.G., ”Modern Methods in Analytical Acoustics”, Lecture Notes, Springer Verlag 1992.

[2] Lighthill, M.J., ”On sound generated aerodynamically. I. General theory”, Proc.Roy.Soc. A 221, 564-87. 1952.

A Averaging with finite sample time

Let us determine the deviation of the approximate time average $\overline{p}^T$, i.e. when using a finite averaging period T, from the true time average $\overline{p}$. The actual pressure is $p = \overline{p} + p'$. Inserting this into the definition of the approximate time average (3) for a windowing function $W = 1$ one obtains:

$$\overline{p}^T = \frac{1}{T}\int_0^T (\overline{p} + p')\, dt = \overline{p} + \frac{1}{T}\int_0^T p'\, dt$$

Now, let us assume p' to be sinusoidal, i.e. in general

$$p' = \hat{p}\cos(\omega t - \varphi)$$

So integration yields

$$\overline{p}^T = \overline{p} + \left[\frac{\sin(\omega T)}{\omega T}\cos\varphi - \frac{\cos(\omega T) - 1}{\omega T}\sin\varphi\right]\hat{p} \tag{A.1}$$

Let us look at the maximum occuring difference between the true and the approximate average $\Delta\overline{p} := \overline{p}^T - \overline{p}$ of all phase shifts φ, i.e.

$$\left.\frac{d\Delta\overline{p}}{d\varphi}\right|_{\max} \stackrel{!}{=} 0 \Leftrightarrow \frac{\sin(\omega T)}{\omega T}\sin\varphi_{\max} + \frac{\cos(\omega T) - 1}{\omega T}\cos\varphi_{\max} \stackrel{!}{=} 0 \Leftrightarrow \cos(\omega T - \varphi_{\max}) \stackrel{!}{=} \cos\varphi_{\max}, \tag{A.2}$$

of which one may easily find that $\varphi_{\max} = \omega T/2$. Using this result in (*A.1*) we obtain the relative error, which $\overline{p}^T$ contains due to the non-perfect averaging-out of the frequency component $f = \omega/(2\pi)$:

$$\Delta\overline{p}_{\max}/\hat{p}(\omega) = -\frac{2\sin(\omega T/2)}{\omega T} \tag{A.3}$$

Note, that for $T \to \infty \Rightarrow \overline{p}^T = \overline{p}$. Equivalently; the higher the frequency ω, the smaller the error in $\overline{p}$ for a given averaging period T.

let us now use a windowing function $W = 1 - \cos(2\pi t/T)$ for the approximate averaging (3). Then the same procedure as above will yield

$$\Delta\overline{p}^W_{\max}/\hat{p}(\omega) = 2\sin(\omega T/2)\left[-\frac{1}{\omega T} + \frac{1}{2(\omega T - 2\pi)} + \frac{1}{2(\omega T + 2\pi)}\right] \tag{A.4}$$

Figure *A.1* shows an example plot of $\Delta\overline{p}_{\max}/\hat{p}$ and $\Delta\overline{p}^W_{\max}/\hat{p}$ for a signal frequency of $f = 16$Hz as a function of the averaging duration T. One may see that for about $T \approx 0.5$s, the error is about 4% with no windowing applied, while it is 0.06% with windowing.

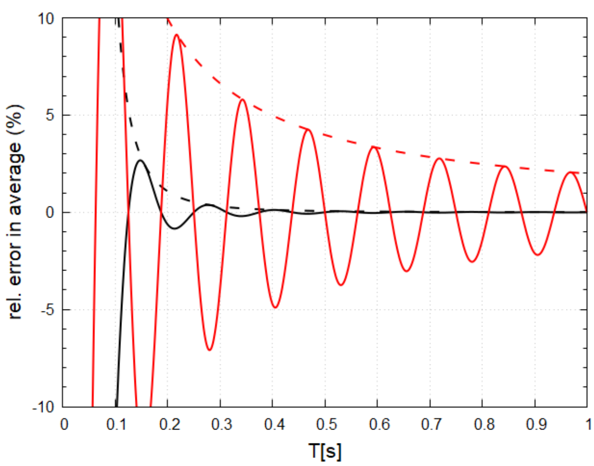

Figure A.1: Relative error in average value in percent due to averaging over finite duration T for a signal frequency of $f = 16$Hz. Red: no window, black: window (see A.4). Dashed curves indicate envelopes to curves

B Free field Green's functions

The 3D Green's function (174) for the convective wave equation in spectral space follows by Fourier-transforming (174):

$$\hat{G}_0^{(3D)} = \frac{\exp(-ikr^+)}{4\pi r^*} \qquad (B.5)$$

with $r^+ = (-rM_r + r^*)/(1-M^2)$ and $r^* = r\sqrt{M_r^2 + 1 - M^2}$.

The 2D and 1D free field Green's function for the convective wave equation (165) can be derived from the 3D free field Green's function (174)

$$G_0^{(2D)} = \frac{H\left(t - \tau + \frac{r}{a_\infty}\frac{M_r - \sqrt{1-M^2+M_r^2}}{1-M^2}\right)}{2\pi\sqrt{1-M^2}\sqrt{\left[t-\tau+\frac{r}{a_\infty}\frac{M_r}{1-M^2}\right]^2 - \frac{r^2}{a_\infty^2}\frac{(1-M^2+M_r^2)}{(1-M^2)^2}}} \qquad (B.6)$$

The corresponding 2D Green's function in the spectral space follows upon Fourier-transforming (*B.6*) in time:

$$\hat{G}_0^{(2D)} = -\frac{i}{4\sqrt{1-M^2}}\exp\left(ikr\frac{M_r}{1-M^2}\right)H_0^{(2)}\left(kr\frac{\sqrt{1-M^2+M_r^2}}{1-M^2}\right) \qquad (B.7)$$

The 1D Green's function is similarly derived from the 3D Green's function by integration over the plane $x = 0$ perpendicular to the flow direction, i.e. $G_0^{(1D)} = \int_0^{2\pi}\int_0^{\infty} G_0 \; R\,dR\,d\varphi$ with $R^2 = (y-\eta)^2 + (z-\zeta)^2$ and φ the circumferential direction around the x-axis (details see appendix):

$$G_0^{(1D)} = \frac{a_\infty}{2}H\left(t - \tau - \frac{r}{a_\infty}\frac{1}{1+M_r}\right) \qquad (B.8)$$

Again, the 1D Green's function in spectral space follows by Fourier transformation in time of G_0^{1D}:

$$\hat{G}_0^{(1D)} = -\frac{i}{2k}\exp\left(-ikr\frac{1}{1+M_r}\right) \qquad (B.9)$$

The table below shows the collection of Green's functions for Poisson's equation, the Helmholtz equation and the wave equation in 1D, 2D and 3D.

equation	1D $\boldsymbol{x} = x$ $\boldsymbol{\xi} = \xi$	2D $\boldsymbol{x} = {}^t(x, y)$ $\boldsymbol{\xi} = {}^t(\xi, \eta)$	3D $\boldsymbol{x} = {}^t(x, y, z)$ $\boldsymbol{\xi} = {}^t(\xi, \eta, \zeta)$
$-\Delta G = \delta(\boldsymbol{x} - \boldsymbol{\xi})$	$-\frac{1}{2} r$	$-\frac{1}{2\pi} \ln r$	$\frac{1}{4\pi r}$
$-\Delta G - k^2 G = \delta(\boldsymbol{x} - \boldsymbol{\xi})$	$-\frac{i}{2k} \exp(-ikr)$	$-\frac{i}{4} H_0^{(2)}(kr)$	$\frac{\exp(-ikr)}{4\pi r}$
$\frac{1}{a_\infty^2} \frac{\partial^2 G}{\partial t^2} - \Delta G = \delta(\boldsymbol{x} - \boldsymbol{\xi}) \delta(t - \tau)$	$\frac{a_\infty}{2} H(t - \tau - \frac{r}{a_\infty})$	$\frac{1}{2\pi} \frac{H(t - \tau - \frac{r}{a_0})}{\sqrt{(t-\tau)^2 - \frac{r^2}{a_\infty^2}}}$	$\frac{\delta(t - \tau - \frac{r}{a_\infty})}{4\pi r}$

<u>Table B.1</u>: Green's functions for various equations and dimensions, whereby the distance r is defined as $r := |\boldsymbol{x} - \boldsymbol{\xi}|$ In the Helmholtz equation the wavenumber is denoted $k := \omega / a_\infty$ and the assumed time factor is $\exp(+i\omega t)$.

C Solutions for point mass or heat source in uniform flow

C.1 Particle velocity for 3D point mass or heat source in uniform flow

According to (177) the acoustic particle velocity of a point source in uniform flow is

$$\begin{aligned} \boldsymbol{v}'(\boldsymbol{x},t) &= \frac{r_0^+}{4\pi\rho^0 a_0 r_0^2(1-M^2\sin^2\theta_0)}\, \boldsymbol{e}_r^+ \frac{\partial\theta_p}{\partial t}\Big|_{t-r_0^+/a_0} + \\ &+ \frac{1}{4\pi\rho^0 r_0^2}\frac{M\cos\theta_0\boldsymbol{M}+(1-M^2)\boldsymbol{e}_r}{\sqrt{1-M^2\sin^2\theta_0}^3}\theta_p\big|_{t-r_0^+/a_0} \end{aligned} \qquad (C.1)$$

Remember that $\boldsymbol{e}_r^+ = \boldsymbol{r}_0/r_0^+ - \boldsymbol{M}$ represents the direction out of which the signal arrives at the observer (propagation direction), i.e. in the farfield the acoustic particle velocity points exactly along the effective propagation direction $\boldsymbol{e}_r^+$ as expected.

C.2 2D acoustic field of line mass or heat source in uniform flow

The acoustic field due to a harmonic line mass or heat source $\frac{D_\infty\dot{\theta}'}{Dt}$ with $\dot{\theta}' = \hat{\theta}_p\exp(i\omega t)\delta(\boldsymbol{x}-\boldsymbol{\xi}_0)$ in two dimensions is

$$\begin{aligned} p'(\boldsymbol{x},t) &= \frac{\omega\hat{\theta}_p}{4\sqrt{1-M^2}^3}\Bigg\{\left[J_0(kr^*)+\frac{M_r}{\sqrt{1-M^2+M_r^2}}Y_1(kr^*)\right]\cos\left(\omega t+kr\frac{M_r}{1-M^2}\right)+ \\ &\qquad \left[Y_0(kr^*)-\frac{M_r}{\sqrt{1-M^2+M_r^2}}J_1(kr^*)\right]\sin\left(\omega t+kr\frac{M_r}{1-M^2}\right)\Bigg\} \\ \rho'(\boldsymbol{x},t) &= \frac{1}{a_\infty^2}p'(\boldsymbol{x},t) \qquad (C.2) \\ \boldsymbol{v}'(\boldsymbol{x},t) &= \frac{-\omega\hat{\theta}_p}{4a_\infty\rho_\infty\sqrt{1-M^2}^3}\Bigg\{\left[\boldsymbol{M}J_0(kr^*)+\frac{(1-M^2)\boldsymbol{e}_r+M_r\boldsymbol{M}}{\sqrt{1-M^2+M_r^2}}Y_1(kr^*)\right]\cos\left(\omega t+kr\frac{M_r}{1-M^2}\right)+ \\ &\qquad \left[\boldsymbol{M}Y_0(kr^*)-\frac{(1-M^2)\boldsymbol{e}_r+M_r\boldsymbol{M}}{\sqrt{1-M^2+M_r^2}}J_1(kr^*)\right]\sin\left(\omega t+kr\frac{M_r}{1-M^2}\right)\Bigg\} \end{aligned}$$

where $r^* = r\sqrt{1-M^2+M_r^2}/(1-M^2)$, in which $r=|\boldsymbol{r}|$ with $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{\xi}_0$ and $M_r = \boldsymbol{r}\cdot\boldsymbol{M}/r$ while $\boldsymbol{e}_r = \boldsymbol{r}/r$.

C.3 1D acoustic field of plane mass or heat source in uniform flow

The acoustic field due to a plane mass or heat source $\frac{D_\infty\dot{\theta}'}{Dt}$ with $\dot{\theta}' = \theta_p(t)\delta(\boldsymbol{x}-\boldsymbol{\xi}_0)$ in one space dimension is

$$p'(x,t) = \frac{a_\infty}{2}\frac{1}{1+M_r}\theta_p\left(t-\frac{r}{a_\infty}\frac{1}{1+M_r}\right)$$

$$\begin{aligned}
\rho'(x,t) &= \frac{1}{a_\infty^2} p'(\boldsymbol{x},t) \\
\boldsymbol{v}'(x,t) &= -\frac{1}{2\rho_\infty}\frac{1}{1+M_r}\theta_p\left(t-\frac{r}{a_\infty}\frac{1}{1+M_r}\right)
\end{aligned} \qquad (C.3)$$

where $r = |\boldsymbol{r}|$ with $\boldsymbol{r} = x - \xi_0$ and $M_r = \boldsymbol{r}\cdot\boldsymbol{M}/r$.

In the spectral space this becomes:

$$\begin{aligned}
\hat{p}(x,\omega) &= \frac{a_\infty}{2}\frac{1}{1+M_r}\exp\left(-ikr\frac{1}{1+M_r}\right)\hat{\theta}_p \\
\hat{\rho}(x,\omega) &= \frac{1}{a_\infty^2}\hat{p}(\boldsymbol{x},\omega) \\
\hat{\boldsymbol{v}}(x,\omega) &= -\frac{1}{2\rho_\infty}\frac{1}{1+M_r}\exp\left(-ikr\frac{1}{1+M_r}\right)\hat{\theta}_p
\end{aligned} \qquad (C.4)$$

D Parallel flow independence on direction of flow

According to the parallel flow assumption $\boldsymbol{v}^0 = \boldsymbol{e}_x u^0$ the x-momentum conservation equation states

$$u^0 \frac{\partial u^0}{\partial x} + \frac{1}{\rho^0}\frac{\partial p^0}{\partial x} = 0. \qquad (D.1)$$

Assuming isentropic variations along the streamlines (along x), i.e. $\rho^0 = \rho_\infty (p^0/p_\infty)^{1/\gamma}$ we therefore have

$$\frac{1}{\rho^0}\frac{\partial p^0}{\partial x} = \frac{p_\infty}{\rho_\infty}\Big(\frac{p^0}{p_\infty}\Big)^{-\frac{1}{\gamma}}\frac{\partial (p^0/p_\infty)}{\partial x} = \frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty}\frac{\partial}{\partial x}\Big[\Big(\frac{p^0}{p_\infty}\Big)^{\frac{\gamma-1}{\gamma}}\Big],$$

which inserted into (*D.1*) leaves

$$\frac{\partial}{\partial x}\Big\{\tfrac{1}{2}u^{02} + \frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty}\Big(\frac{p^0}{p_\infty}\Big)^{\frac{\gamma-1}{\gamma}}\Big\} = 0$$

or accordingly the statement, that the term to be differentiated w.r.t. x is a constant $B \neq B(x)$, but generally dependent on y, z:

$$\tfrac{1}{2}u^{02} + \frac{\gamma}{\gamma-1}\frac{p_\infty}{\rho_\infty}\Big(\frac{p^0}{p_\infty}\Big)^{\frac{\gamma-1}{\gamma}} \stackrel{!}{=} B(y,z)$$

solved for the pressure or density respectively we obtain

$$\frac{p^0}{p_\infty} = \Big[(B - \tfrac{1}{2}u^{02})\frac{\gamma-1}{\gamma}\frac{\rho_\infty}{p_\infty}\Big]^{\frac{\gamma}{\gamma-1}} \Rightarrow \frac{\rho^0}{\rho_\infty} = \Big[(B - \tfrac{1}{2}u^{02})\frac{\gamma-1}{\gamma}\frac{\rho_\infty}{p_\infty}\Big]^{\frac{1}{\gamma-1}} \qquad (D.2)$$

Again, according to the parallel flow assumption $\boldsymbol{v}^0 = \boldsymbol{e}_x u^0$ the mass conservation equation states

$$\frac{\partial}{\partial x}\Big(\frac{\rho^0}{\rho_\infty}u^0\Big) = \frac{\partial}{\partial x}\Big(\frac{\rho^0}{\rho_\infty}\Big)u^0 + \frac{\rho^0}{\rho_\infty}\frac{\partial u^0}{\partial x} = 0$$

which upon inserting ρ^0/ρ_∞ from eqn (*D.2*) gives

$$\frac{1}{\gamma-1}\frac{\gamma-1}{\gamma}\frac{\rho_\infty}{p_\infty}\Big(-u^0\frac{\partial u^0}{\partial x}\Big)\Big[(B-\tfrac{1}{2}u^{02})\frac{\gamma-1}{\gamma}\frac{\rho_\infty}{p_\infty}\Big]^{\frac{2-\gamma}{\gamma-1}}u^0 + \Big[(B-\tfrac{1}{2}u^{02})\frac{\gamma-1}{\gamma}\frac{\rho_\infty}{p_\infty}\Big]^{\frac{1}{\gamma-1}}\frac{\partial u^0}{\partial x} = 0.$$

This may finally be rearranged to

$$\frac{\rho_\infty}{\gamma p_\infty}\Big[(\gamma-1)B - \frac{\gamma+1}{2}u^{02}\Big]\frac{\partial u^0}{\partial x} = 0.$$

Since $B \neq B(x)$, the bracket expression cannot be zero for u^0 varying with x. Therefore, necessarily $u^0 \neq u^0(x)$.

E Nearfield terms in FW-H integral

$$
\begin{aligned}
T_{nf} \quad = \quad & \frac{a_\infty}{r^2}\Big[-\frac{\dot{T}_{kk}}{(1-M_r)^2} - \frac{4M\dot{T}_{rM} + T_{kk}\dot{M}_r + 2\dot{M}T_{r\dot{M}}}{(1-M_r)^3} + \\
& + \frac{3(1-M^2)\dot{T}_{rr} - 6MT_{rM}\dot{M}_r - T_{rr}\dot{\boldsymbol{M}}\cdot\boldsymbol{M}}{(1-M_r)^4} + \frac{3T_{rr}\dot{M}_r(1-M^2)}{(1-M_r)^5}\Big] + \\
& + \frac{a_\infty^2}{r^2}\Big[\frac{-2T_{rr}\dot{\boldsymbol{M}}\cdot\boldsymbol{M}}{(1-M_r)^4} + \frac{3T_{rr}\dot{M}_r(1-M^2)}{(1-M_r)^5}\Big] + \\
& + \frac{a_\infty^2}{r^3}\Big[\frac{2T_{MM}M^2 - T_{kk}(1-M^2)}{(1-M_r)^3} - \frac{6MT_{rM}\dot{M}_r}{(1-M_r)^4} + \frac{3T_{rr}(1-M^2)^2}{(1-M_r)^5}\Big]
\end{aligned}
$$

$$
f_{nf} = \frac{a_\infty}{r^2}\Big[\frac{-f_M}{(1-M_r)^2} + \frac{f_r(1-M^2)}{(1-M_r)^3}\Big]
$$

$$
m_{nf} = \frac{a_\infty}{r^2}\Big[\frac{m_n(M_r - M^2)}{(1-M_r)^3}\Big]
$$

F Example Problems

F.1 Sound of a bursting bubble

A soap bubble of Radius R bursts at $t = 0$. The soap skin has a surface tension of T. Determine the sound field generated after the burst.

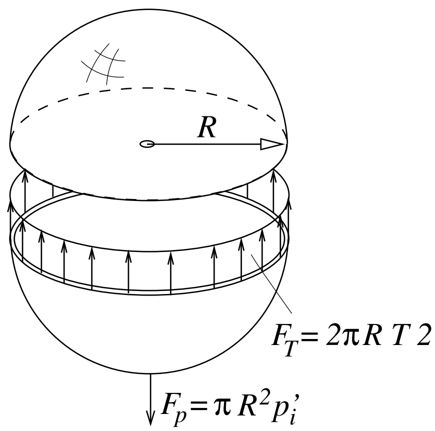

Figure F.1: A soap bubble before burst.

Prior to the burst there exists an interior overpressure of p_i'. Cutting the spherical bubble into two equal halfs allows to apply force equilibrium in order to find the value of that pressure. The surface tension is by definition a line force, which is constant along the circumference of the cut bubble surface. The overall surface tension force F_T balancing the pressure force F_p, is therefore $(2\pi R) \cdot T \cdot 2$. Note the occurence of the factor 2 because the skin of the bubble consists of two surfaces (the inner and the outer surface). The pressure force $F_p = \pi R^2 \cdot p_i'$. Therefore the pressure in the bubble is:

$$p_i' = 4\frac{T}{R}$$

F.1.1 Approach

We describe the acoustic field using a potential φ'. The potential satisfies the same wave equation as the pressure p'. Therefore we start with assuming the general solution according to d'Alembert:

$$\varphi' = \frac{f(t - r/a_0) + g(t + r/a_0)}{r}$$

F.1.2 Boundary conditions

Since the problem is symmetric about the centre $r = 0$ the acoustic particle velocity contains only a radial component: $\boldsymbol{v}' = v'\boldsymbol{e}_r$. The only boundary condition we may assume is at $r = 0$. For reasons of symmetry the particle velocity $\boldsymbol{v}'$ at $r = 0$ has to vanish for all times t:

$$v'(r = 0, t) = \boldsymbol{\nabla}\varphi' \cdot \boldsymbol{e}_r = \frac{\partial \varphi'}{\partial r} = 0.$$

according to (64). The particle velocity is

$$v'(r,t) = \frac{-f'(t - r/a_0) + g'(t + r/a_0)}{a_0 r} - \frac{f(t - r/a_0) + g(t + r/a_0)}{r^2}$$

Note that for $r \to 0$ this is an undetermined expression. In order to apply the condition one has to use Hopital's rule, which requires first to re-arrange v' into an expression of type "0/0" near $r = 0$:

$$v'(r,t) = \frac{r[-f'(t - r/a_0) + g'(t + r/a_0)] - a_0[f(t - r/a_0) + g(t + r/a_0)]}{a_0 r^2}$$

Hopital's rule yields:

$$\lim_{r\to 0} v' = \lim_{r\to 0} \left\{ \frac{r[f''(t - r/a_0) + g''(t + r/a_0)]}{2a_0 r} \right\} = 0$$

which immediately gives

$$g''(t) = -f''(t) \Longrightarrow g(t) = -f(t) + C_1 t + C_0$$

Since the argument t is not restricted by whatsoever condition it may assume any value. Therefore one may also replace t by $t + r/a_0$ to arrive at the general form of g

$$g(t + r/a_0) = -f(t + r/a_0) + C_1(t + r/a_0) + C_0$$

F.1.3 Initial conditions

Prior to the burst there is no motion anywhere. For all r at $t = 0$ we have

$$\rho^0 a_0 r^2 v'(r, t = 0) = r[-f'(-r/a_0) - f'(r/a_0) + C_1] - a_0[f(-r/a_0) - f(r/a_0) + C_1 r/a_0 + C_0] = 0$$

The only way this equation can be satisfied for all r is when

$$f'(-r/a_0) = -f'(r/a_0) \Longrightarrow f(-r/a_0) = f(r/a_0)$$

i.e. for f being an even function and $C_0 = 0$.

There exists an initial condition for the pressure perturbation p' as well. First we express p' in terms of the potential and the above results, eqn (66):

$$p'(r,t) = -\rho^0 \frac{\partial \varphi'}{\partial t} = -\frac{\rho^0}{r}[f'(t - r/a_0) - f'(t + r/a_0) + C_1]$$

For $t = 0$ this can be re-arranged to

$$p'(r, t = 0) = -\frac{\rho^0}{r}[\underbrace{f'(-r/a_0) - f'(r/a_0)}_{= -2f'(r/a_0)} + C_1]$$

The pressure inside the buble is $p' = p'_i = 4T/R$ and outside $p' = 0$:

$$p'(r, t=0) = \frac{\rho^0}{r}[2f'(r/a_0) - C_1] = \begin{cases} 4T/R & \text{for } 0 < r < R \\ 0 & \phantom{\text{for }} r > R \end{cases}$$

This gives the following determining equation for f':

$$\begin{aligned} f'(r/a_0) &= \begin{cases} 2\frac{T}{R}\frac{a_0}{\rho^0}\frac{r}{a_0} + \frac{1}{2}C_1 & \text{for } 0 < r < R \\ \frac{1}{2}C_1 & \phantom{\text{for }} r > R \end{cases} \\ &= -f'(-r/a_0) \Longrightarrow C_1 = 0 \end{aligned}$$

The conclusion that $C_1 = 0$ follows from the fact that f' is an odd function.

F.1.4 General solution

We may now generalize the above result for f':

$$f'(t - r/a_0) = \begin{cases} 2\frac{T}{R}\frac{a_0}{\rho^0}(t - r/a_0) & \text{for } 0 < t - r/a_0 < R/a_0 \quad \text{or} \quad 0 < -t + r/a_0 < R/a_0 \\ 0 & \phantom{\text{for }} t - r/a_0 > R/a_0 \quad \text{or} \quad -t + r/a_0 > R/a_0 \end{cases}$$

$$f'(t + r/a_0) = \begin{cases} 2\frac{T}{R}\frac{a_0}{\rho^0}(t + r/a_0) & \text{for } 0 < t + r/a_0 < R/a_0 \quad \text{or} \quad 0 < -t - r/a_0 < R/a_0 \\ 0 & \phantom{\text{for }} t + r/a_0 > R/a_0 \quad \text{or} \quad -t - r/a_0 > R/a_0 \end{cases}$$

The respective domains where $f' \neq 0$ are marked in the figure *F.2*. Obviously the pressure p' is of interest only for $t \geq 0$. In order to simplify the discussion of the solution let us assume to first look only at positions outside the original radius of the bubble: $r > R$. Then $t + r/a_0 > R/a_0 \Longrightarrow f'(t + r/a_0) = 0$, i.e. the advanced time part of the solution vanishes here. Then we have

$$p'(r, t) = \begin{cases} \frac{2T}{R}(1 - \frac{a_0 t}{r}) & \text{for } 0 < |t - r/a_0| < R/a_0 \\ 0 & \phantom{\text{for }} |t - r/a_0| > R/a_0 \end{cases}$$

For a fixed position r the time signal is a (piecewise) linear function, while for a fixed time (a shapshot) the pressure decays like a (piecewise) hyperbola with the distance from the origin. The figure *F.3* shows the time and space signature of the resulting pressure pulse, forming a so called N-wave (according to the shape of the signal) in time.

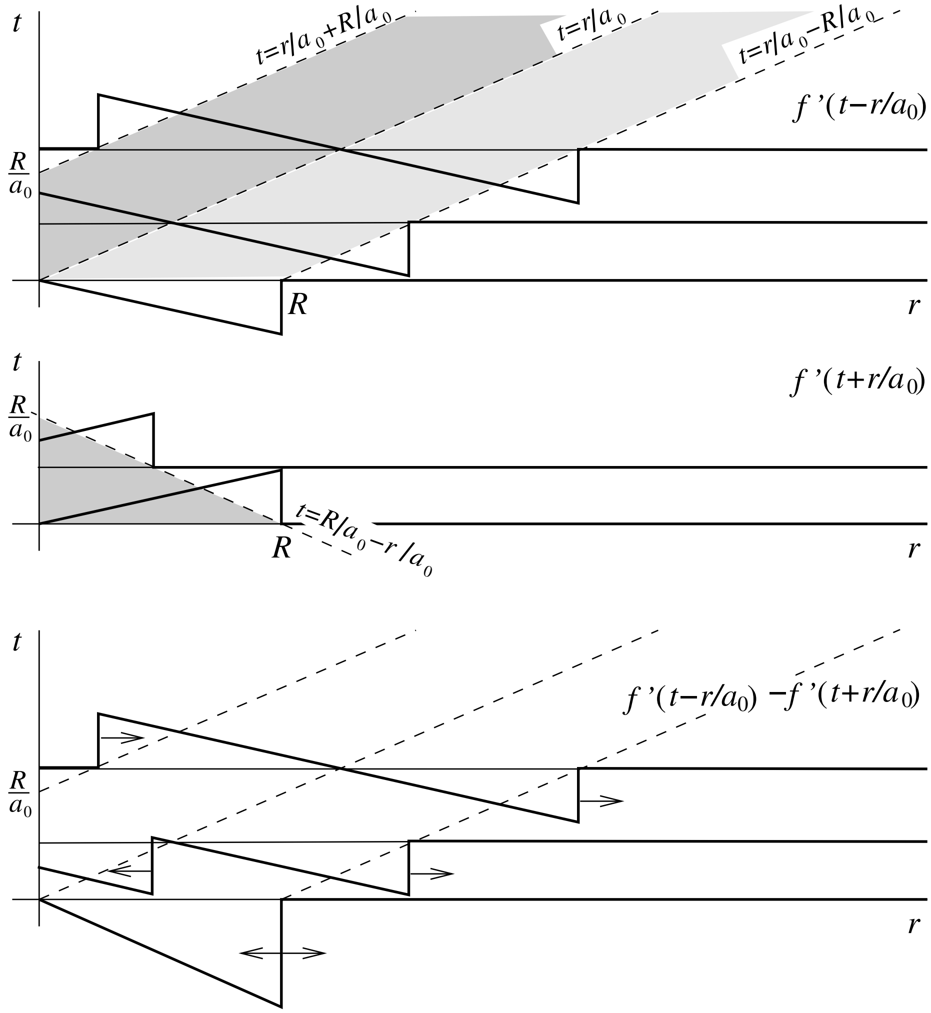

Figure F.2: Functions $f'(t - r/a_0)$ and $f'(t + r/a_0)$.

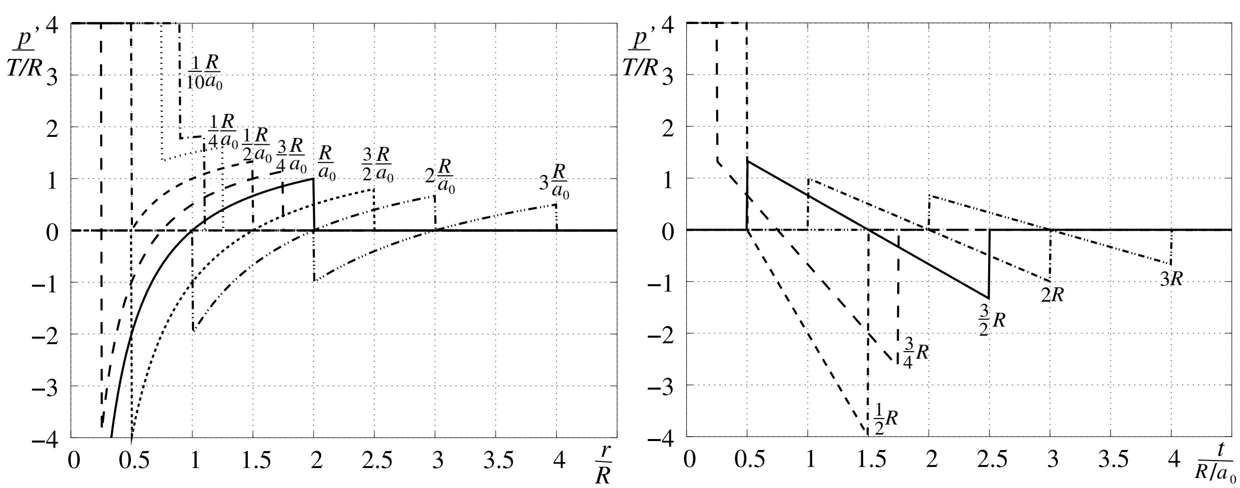

Figure F.3: Pressure pulse. Left: spatial signature for different times, right: temporal signature for different positions.

F.2 Sound of a harmonically pulsating ("breathing") sphere

A sphere oscillates radially about its average radius R with a small amplitude $A_0 \ll R$ at a circular frequency of Ω. What sound field is generated?

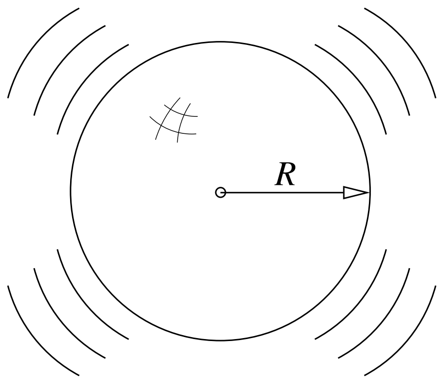

Figure F.4: A pulsating impermeable sphere surface.

The sphere's surface moves like

$$\delta(t) = A_0 \exp(i\Omega t), \quad \delta \ll R$$

in the radial direction. The respective radial velocity of the sphere is therefore

$$v_r'(r = R) = \frac{\partial \delta}{\partial t} = i\Omega A_0 \exp(i\Omega t)$$

We use the velocity potential φ' to describe the sound field. The general form of the solution (satisfying the radiation condition at infinity) is

$$\varphi'(r, t) = \varphi_0 \frac{1}{r} \exp(i\omega t - ikr)$$

For reasons of symmetry the acoustic particle velocity has only a radial component, which may be computed from the potential through (64):

$$v' = \frac{\partial \varphi'}{\partial r} = \frac{\varphi_0}{r}(-ik - \frac{1}{r}) \exp(i\omega t - ikr)$$

The velocity of the medium at the sphere's surface has to be the same as that of the surface

$$\frac{\varphi_0}{R}(-ik - \frac{1}{R}) \exp(i\omega t) \exp(-ikR) = i\Omega A_0 \exp(i\Omega t)$$

This equality can only hold for:

$$\omega = \Omega \quad \text{and} \quad \varphi_0 = i\Omega \frac{R}{-ik - 1/R} \exp(-ikR) A_0$$

Having determined the potential, we may now compute the pressure according to (66)

$$\begin{aligned} p' &= -\Omega^2 \frac{A_0}{R} \rho^0 \frac{R^2(1 - ikR)}{1 + (kR)^2} \frac{R}{r} \exp[i\Omega t - ik(r - R)] \\ &= -\rho^0 a_0^2 \frac{\delta(t)}{R} \frac{(kR)^2}{1 + (kR)^2} (1 - ikR) \frac{R}{r} \exp[-ik(r - R)] \end{aligned}$$

where $k = a_0\Omega = 2\pi/\lambda$. The solution shows that for small radii of the sphere (compared to a wavelength of the radiated sound) $R/\lambda \ll 1$ the pressure amplitude decreases, i.e. a small sphere generates sound only inefficiently compared to a large sphere.

Note that if φ' is a solution of the wave equation, then $\dfrac{\partial\varphi'}{\partial\tilde{x}} = \boldsymbol{e}_{\tilde{x}}\boldsymbol{\cdot}\boldsymbol{\nabla}\varphi'$, where $\tilde{x} := e_{\tilde{x}}\boldsymbol{\cdot}\boldsymbol{x}$ is a solution to the wave equation as well.

F.3 Sound of a harmonically oscillating sphere

A rigid sphere of radius R oscillates harmonically about its average centre along a direction $\boldsymbol{e}_{\tilde{x}}$ with a small amplitude $A_0 \ll R$ at a circular frequency of Ω. What sound field is generated?

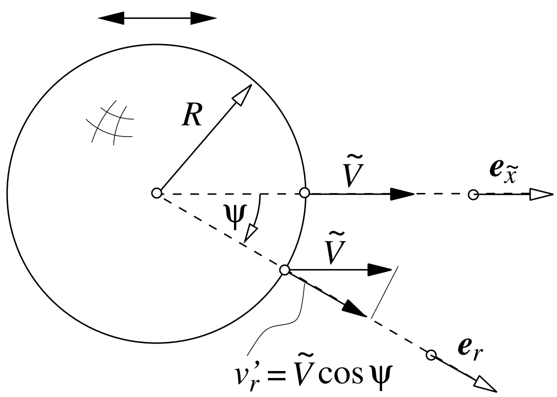

Figure F.5: An oscillating impermeable rigid sphere surface.

The sphere's surface moves like

$$\delta(t) = A_0 \exp(i\Omega t), \quad \delta \ll R$$

in the direction $\boldsymbol{e}_{\tilde{x}}$. The respective velocity of the sphere in this direction is therefore

$$v'_{\tilde{x}}(r = R) = \frac{\partial \delta}{\partial t} = i\Omega A_0 \exp(i\Omega t) = i\Omega\delta(t) = \tilde{V} \exp(i\Omega t)$$

Adjacent to it's surface the sphere displaces fluid according to the velocity component along it's normal. Therefore the radial component of the surface velocity is important:

$$v'_r(r = R) = \cos\psi \; v'_{\tilde{x}}(r = R) = \cos\psi \; \tilde{V} \exp(i\Omega t)$$

We start with the velocity potential φ' to describe the sound field. Certainly, we cannot use our spherically symmetric harmonic solution $\varphi' = \phi_0 \exp(\omega t - \underbrace{\omega/a_0}_{=:\,k} r)/r$ to solve this problem. But from eqn (93) we know that we may generate new solutions to the wave equation by taking the derivative of known solutions. One possibility is to differentiate φ' by $\tilde{x}$ and try to satisfy the boundary conditions at $r = R$ with this new elementary solution:

$$\tilde{\varphi}' = \tilde{\varphi}_0 \frac{1}{r}(-ik - 1/r)\frac{\partial r}{\partial \tilde{x}} \exp(i\omega t - ikr)$$

Now

$$\frac{\partial r}{\partial \tilde{x}} = \boldsymbol{e}_{\tilde{x}} \cdot \underbrace{\boldsymbol{\nabla} r}_{\boldsymbol{e}_r} = \boldsymbol{e}_{\tilde{x}} \cdot \boldsymbol{e}_r = \cos\psi$$

according to figure F.5. From the general definition of $\boldsymbol{v}' = \boldsymbol{\nabla}\tilde{\varphi}'$ the required radial velocity field at $r = R$ is found to be

$$v'_r(r = R) = \frac{\partial \tilde{\varphi}'}{\partial r} = \tilde{\varphi}_0 \left[\frac{2}{R^3} + \frac{2ik}{R^2} - \frac{k^2}{R}\right] \exp(i\omega t) \exp(-ikR) \cos\psi$$

The comparision with the above mentioned boundary conditions of the problem shows that the direction cosine $\cos\psi$ drops out and that the remaining terms may indeed be matched with

$$\omega = \Omega \quad , \quad \tilde{\varphi}_0 = i\Omega A_0 \exp(ikR) / \left[\frac{2}{R^3} + \frac{2ik}{R^2} - \frac{k^2}{R}\right]$$

with the velocity potential $\tilde{\varphi}'$ thus known, it is easy to determine the pressure field from (66)

$$\begin{aligned} p' = -\rho^0 \frac{\partial \tilde{\varphi}'}{\partial t} &= -\rho^0 a_0^2 \cos\psi \frac{A_0}{R} \frac{(kR)^2}{2 + 2ikR - (kR)^2} \frac{R}{r} \left[\frac{R}{r} + ikR\right] \exp[i\Omega t - ik(r - R)] \\ &= -\rho^0 a_0^2 \cos\psi \frac{\delta(t)}{R} \frac{(kR)^2}{2 + 2ikR - (kR)^2} \frac{R}{r} \left[\frac{R}{r} + ikR\right] \exp[-ik(r - R)] \end{aligned}$$

where $k = a_0\Omega = 2\pi/\lambda$. The solution shows that the pressure behaves differently near and far from the sphere's surface $r = R$. There is a part, decaying like R/r and one decaying more rapidly like R^2/r^2. The former is also called *farfield* (german: "Fernfeld"), the latter is also called *nearfield* (german: "Nahfeld"). Note that such a nearfield does not exist in the pressure field of the pulsating sphere (Example 2).

Next observe that the amplitude of the sound field is depending on the angle ψ. This directivity is such that along $\psi = 0$ and $\psi = \pi$ or $\pm\boldsymbol{e}_{\tilde{x}}$ most sound is radiated, while under $\psi = \pi/2$ exactly no sound is radiated. Such a sound field is called to have dipole character.

Also note that the solution for the oscillating sphere would have resulted from the solution of the pulsating sphere p'_{ps} (Example 2) basically by differentiating the latter with respect to the direction $\tilde{x}$

$$p' = -\frac{(1 + ikR)R}{2 + 2ikR - (kR)^2} \frac{\partial p'_{ps}}{\partial \tilde{x}}$$

F.4 Scattering of plane wave at cylinder

The scattering of a plane Gaussian-shaped acoustic pressure pulse p' at a cylinder with radius R and center at $x = 0, y = 0$ is considered. The initial condition is:

$$p'(t = 0, x, y) = \exp\left(-\frac{(x - x_s)^2}{h_w^2} \ln 2\right) \tag{F.1}$$

where h_w denotes the half-width of the Gaussian function and x_s is the initial position of the center of the Gaussian. The pressure must satisfy the hard wall boundary condition at the wall of the cylinder, easiest expressed in an axi-symmetric co-ordinate system (r, ϑ) with $x = r\cos\vartheta$ and $y = r\sin\vartheta$:

$$\frac{\partial p'}{\partial r}(t, r = R, \vartheta) = 0 \tag{F.2}$$

The governing wave equation for the pressure field is:

$$\frac{\partial^2 p'}{\partial t^2} - c^2 \left(\frac{\partial^2 p'}{\partial r^2} + \frac{1}{r}\frac{\partial p'}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p'}{\partial \vartheta^2}\right) = 0 \tag{F.3}$$

The pressure field $p' = p_p + p_s$ may be split into a primary wave $p_p(t, x, y)$ and a scattered wave $p_s(t, r, \vartheta)$. The primary wave p_p is assumed to behave as if the cylinder was absent, i.e. a plane pulse, traveling with the speed of sound c:

$$p_p(t, x, y) = \exp\left(-\frac{(x - x_s - ct)^2}{h_w^2} \ln 2\right) = \exp\left(-\frac{(r\cos\vartheta - c\tau)^2}{h_w^2} \ln 2\right) \tag{F.4}$$

where the shifted time

$$\tau = t + x_s/c \tag{F.5}$$

was introduced for convenience. The scattered wave p_s is computed according to (*F.3*) under the (known) inhomogeneous boundary condition

$$\frac{\partial p_s}{\partial r}(\tau, r = R, \vartheta) = -\frac{\partial p_p}{\partial r}(\tau, r = R, \vartheta) \tag{F.6}$$

Since the fields are single valued one may expand them into a Fourier series in ϑ and then consider its coefficients $m = 0, \ldots, \infty$ separately:

$$p(\tau) = \sum_{0}^{\infty} p_m(\tau)\cos(m\vartheta) \tag{F.7}$$

where explicit use has been made of the fact that the fields are symmetric w.r.t. the x-axis (vanishing of sine part). The problem becomes even more explicit in the frequency domain. Upon Fourier cosine and sine transforming (*F.3*) in time τ one obtains the mth order Bessel equation, governing the mth Fourier coefficient $\hat{p}_m(\omega, r)$ of the Fourier series in ϑ:

$$\left[\left(\frac{\omega}{c}\right)^2 - \frac{m^2}{r^2}\right]\hat{p}_m + \frac{d^2\hat{p}_m}{dr^2} + \frac{1}{r}\frac{d\hat{p}_m}{dr} = 0 \tag{F.8}$$

where $\hat{p}_m$ stands for both the cosine- and sine coefficients $\hat{p}^c_m(\omega)$ and $\hat{p}^s_m(\omega)$ defined as:

$$\hat{p}^c_m(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} p_m(\tau)\cos(\omega\tau)d\tau \quad , \quad \hat{p}^s_m(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} p_m(\tau)\sin(\omega\tau)d\tau \qquad (F.9)$$

$$p_m(\tau) = \int_0^{\infty} \hat{p}^c_m(\omega)\cos(\omega\tau) \quad + \quad \hat{p}^s_m(\omega)\sin(\omega\tau)\, d\omega \quad \text{(inversion)} \qquad (F.10)$$

The boundary condition (*F.6*) now reads:

$$\frac{(d\hat{p}_s)_m}{dr}(\omega, r = R) = -\frac{(d\hat{p}_p)_m}{dr}(\omega, r = R) \qquad (F.11)$$

The general solution of (*F.8*) is

$$\hat{p}_m = C_m(k)J_m(kr) + D_m(k)Y_m(kr) \qquad (F.12)$$

where $k = \omega/c$ is the wave number, J_m and Y_m represent the Bessel functions of order m of the first and second kind respectively (see e.g.). Further, $C_m(k)$ and $D_m(k)$ denote coefficients, depending on k. It is these coefficients which are to be determined as to let $p_s + p_p$ satisfy the boundary conditions (*F.11*) at $r = R$ and the condition of outgoing waves for p_s.

The implementation of (*F.11*) necessitates the Fourier transform in time of $p_p(\tau)$:

$$\hat{p}^c_p(\omega) = F(k)\cos(\omega x/c) \ , \quad \hat{p}^s_p(\omega) = F(k)\sin(\omega x/c) \qquad (F.13)$$

with the abbreviation

$$F(k) = \frac{h_w}{c}\sqrt{\frac{1}{\pi\ln 2}}\exp\Big(-k^2\frac{h_w^2}{4\ln 2}\Big) \qquad (F.14)$$

Next, cosine and sine are expressed in terms of the Bessel functions:

$$\begin{aligned} \cos(kx) &= \cos(kr\cos\vartheta) = J_0(kr) + 2\sum_{n=1}^{\infty}(-1)^n\cos(2n\vartheta)J_{2n}(kr) \\ \sin(kx) &= \sin(kr\cos\vartheta) = -2\sum_{n=1}^{\infty}(-1)^n\cos((2n-1)\vartheta)J_{2n-1}(kr) \end{aligned} \qquad (F.15)$$

The respective substitution into (*F.13*) now immediately yields the Fourier coefficients $n = 1, \ldots, \infty$ of the primary wave:

$$\begin{aligned} (\hat{p}^c_p)_0 = F(k)J_0(kr)\ , \quad (\hat{p}^c_p)_{2n} &= 2F(k)(-1)^nJ_{2n}(kr)\ , \\ (\hat{p}^s_p)_{2n-1} &= -2F(k)(-1)^nJ_{2n-1}(kr) \end{aligned} \qquad (F.16)$$

The coefficients $(\hat{p}^c_p)_{2n-1} = (\hat{p}^s_p)_{2n} = (\hat{p}^s_p)_0 = 0$. Upon using the differentiation rule

$$\frac{dJ_m(\eta)}{d\eta} = [J_{m-1}(\eta) - J_{m+1}(\eta)]/2 \quad \text{for } m > 0 \quad \text{and} \quad \frac{dJ_0(\eta)}{d\eta} = -J_1(\eta) \qquad (F.17)$$

one finally obtains for the derivative of the non-zero coefficients of the primary wave $(\hat{p}_p)_m$ at the cylinder surface $r = R$:

$$\begin{aligned} \frac{d}{dr}(\hat{p}^c_p)_0 = -kF(k)[J_1]_{kR},\ \frac{d}{dr}(\hat{p}^c_p)_{2n} &= kF(k)(-1)^n[J_{2n-1} - J_{2n+1}]_{kR} \\ \frac{d}{dr}(\hat{p}^s_p)_{2n-1} &= -kF(k)(-1)^n[J_{2n-2} - J_{2n}]_{kR} \end{aligned} \qquad (F.18)$$

Next, the boundary condition (*F.11*) requires the evaluation of the gradient of the scattered wave Fourier coefficients $(\hat{p}_s)_m$. The differentiation rule (*F.17*) is valid for the functions Y_m too and its application on (*F.12*) leaves:

$$
\begin{aligned}
\frac{d}{dr}(\hat{p}_s)_0^{c/s} &= -kC_0^{c/s}J_1 - kD_0^{c/s}Y_1 \\
\frac{d}{dr}(\hat{p}_s)_m^{c/s} &= kC_m^{c/s}(J_{m-1} - J_{m+1})/2 + kD_m^{c/s}(Y_{m-1} - Y_{m+1})/2
\end{aligned}
\qquad (F.19)
$$

where $(\hat{p}_s)_m^{c/s}$ stands for the two temporal Fourier coefficients, i.e. the cosine $(\hat{p}_s)_m^c$ and sine $(\hat{p}_s)_m^s$ parts. The matching of (*F.19*) with (*F.18*) according to (*F.11*) now yields the first two equations for $C_m^{c/s}, D_m^{c/s}$:

$$
\begin{aligned}
\sin(\gamma_0)\ C_0^c + \cos(\gamma_0)\ D_0^c &= -F(k)\sin(\gamma_0) \\
\sin(\gamma_{2n})\ C_{2n}^c + \cos(\gamma_{2n})\ D_{2n}^c &= -2(-1)^n F(k)\sin(\gamma_{2n}) \\
\sin(\gamma_{2n+1})\ C_{2n+1}^s + \cos(\gamma_{2n+1})\ D_{2n+1}^s &= 2(-1)^n F(k)\sin(\gamma_{2n+1})
\end{aligned}
\qquad (F.20)
$$

where

$$
\begin{aligned}
\sin(\gamma_0) &= \left[\frac{J_1}{\sqrt{J_1^2 + Y_1^2}}\right]_{kR}, \cos(\gamma_0) = \left[\frac{Y_1}{\sqrt{J_1^2 + Y_1^2}}\right]_{kR} \\
\sin(\gamma_m) &= \left[\frac{J_{m-1} - J_{m+1}}{\sqrt{(J_{m-1} - J_{m+1})^2 + (J_{m-1} - J_{m+1})^2}}\right]_{kR} \\
\cos(\gamma_m) &= \left[\frac{Y_{m-1} - Y_{m+1}}{\sqrt{(J_{m-1} - J_{m+1})^2 + (J_{m-1} - J_{m+1})^2}}\right]_{kR}
\end{aligned}
\qquad (F.21)
$$

The equation system for the C_m, D_m is closed by application of the condition of outgoing wave behavior at large kr. Far out, the Bessel function $J_m(kr) \longrightarrow \cos(kr + \phi_m)const/\sqrt{kr}$ and $Y_m(kr) \longrightarrow \sin(kr + \phi_m)const/\sqrt{kr}$. The integrand in (*F.10*) for p_s, i.e.

$$
\begin{aligned}
\lim_{kr\to\infty} \hat{p}_s(r,t)/\sqrt{kr} \sim \quad & [C_m^c \cos(kr + \phi_m) + D_m^c \sin(kr + \phi_m)] \cos(\omega\tau) + \\
& [C_m^s \cos(kr + \phi_m) + D_m^s \sin(kr + \phi_m)] \sin(\omega\tau)
\end{aligned}
\qquad (F.22)
$$

will have to consist of strictly outgoing components, i.e. only combinations $\cos(kr+\phi_m-\omega\tau) = \sin(kr+\phi_m)\sin(\omega\tau)+\cos(kr+\phi_m)\cos(\omega\tau)$ and $\sin(kr+\phi_m-\omega\tau) = \sin(kr+\phi_m)\cos(\omega\tau) - \cos(kr+\phi_m)\sin(\omega\tau)$ are allowed in (*F.22*) leading to the auxiliary relations:

$$
D_m^s - C_m^c = 0 \quad \text{and} \quad C_m^s + D_m^s = 0 \qquad (F.23)
$$

The solution of the system (*F.20*,*F.23*) and substitution into (*F.12*) finally yields $(n = 1, \ldots, \infty)$:

$$
\begin{aligned}
(\hat{p}_s)_0^c &= -F(k)\sin(\gamma_0)\,[\sin(\gamma_0)J_0(kr) + \cos(\gamma_0)Y_0(kr)] \\
(\hat{p}_s)_{2n}^c &= -2F(k)(-1)^n \sin(\gamma_{2n})\,[\sin(\gamma_{2n})J_{2n}(kr) + \cos(\gamma_{2n})Y_{2n}(kr)] \\
(\hat{p}_s)_{2n-1}^c &= 2F(k)(-1)^n \sin(\gamma_{2n-1})\,[\cos(\gamma_{2n-1})J_{2n-1}(kr) - \sin(\gamma_{2n-1})Y_{2n-1}(kr)] \\
& \qquad\qquad (F.24) \\
(\hat{p}_s)_0^s &= F(k)\sin(\gamma_0)\,[\cos(\gamma_0)J_0(kr) - \sin(\gamma_0)Y_0(kr)] \\
(\hat{p}_s)_{2n}^s &= 2F(k)(-1)^n \sin(\gamma_{2n})\,[\cos(\gamma_{2n})J_{2n}(kr) - \sin(\gamma_{2n})Y_{2n}(kr)] \\
(\hat{p}_s)_{2n-1}^s &= 2F(k)(-1)^n \sin(\gamma_{2n-1})\,[\sin(\gamma_{2n-1})J_{2n-1}(kr) + \cos(\gamma_{2n})Y_{2n}(kr)]
\end{aligned}
$$

The $(\hat{p}_s)^c_m$ and $(\hat{p}_s)^s_m$ are respectively the cosine and sine coefficients for the Fourier inversion (*F.10*) back into the time domain $(p_s)_m(\tau = t + x_s)$. The summation over all m in (*F.7*) finally gives the pressure field of the scattered wave p_s.