Precision-Aided Partial Ambiguity Resolution Scheme for GNSS Attitude Determination

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***Abstract*—The use of carrier phase data play an important role for high-precision Global Navigation Satellite Systems (GNSS) positioning solutions, such as Real-Time Kinematic (RTK). Similarly, precise orientation information can be obtained with multi-antenna setups which exploit carrier phase observables. The availability of high precision navigation solutions is, however, subject to the Integer Ambiguity Resolution (IAR) performance. IAR is the process of mapping the real-valued carrier ambiguities to integer ones, enhancing the attitude solution by virtue of the cross-correlation with the estimated integer ambiguities. Unfortunately, IAR is known to suffer from dimensionality course or, in other words, the chances for finding the correct vector of integers reduces with the number of ambiguities.**

This work focuses on improving the availability of high precision attitude estimates by means of using a Partial Ambiguity Resolution (PAR) scheme. PAR relaxes the condition of estimating the complete vector of ambiguities and, instead and finds a subset of them to maximize the availability. A new formulation for attitude determination using quaternion rotation within a precision-driven PAR scheme is proposed. Numerical simulations are used to showcase the attitude determination performance with a conventional Full Ambiguity Resolution (FAR) and a precision-aided PAR approach.

Keywords - GNSS; PAR; Ambiguity Resolution; Attitude Estimation; PNT; GNSS Attitude Model.

I. Introduction

Global Navigation Satellite Systems (GNSS) play a key role for robotic and vehicular applications by providing all-weather, all-time positioning information. The use of carrier phase observations is a main factor for precise navigation, since their noise is two orders of magnitude lower than for code observations. However, carrier phase observations are ambiguous by an unknown integer number of cycles. The process of determining the ambiguities is known as Integer Ambiguity Resolution (IAR), which grants an estimate with high precision.

Attitude determination is also a practical application that involves carrier phase measurements to estimate the orientation of a body with respect to to its environment. In a multi-GNSS system, the rotation estimation relates the baseline vectors to each pair of antenna positions across two frames. An overview of the GNSS-based attitude determination was introduced in [1, Ch. 27] showing the relationship between IAR and the corresponding rotation matrix. The rotation operation can be represented with rotation matrices and quaternions [2]. The quaternion parametrization is considered in this work since the attitude model is generally expressed as a Least Square (LS) adjustment and the quaternion rotation is free of singularities.

With the deployment of new GNSS constellations and frequencies, the large number of observations available can reduce the probability of a correct IAR (i.e., the *success rate* decreases). This phenomenon becomes more accentuated for the GNSS-based attitude model, since the volume of measurements is considerably greater than for positioning problems. Partial Ambiguity Resolution (PAR) is an useful technique in order to increase such success rate [3]–[6]. Thus, PAR relaxes the condition of estimating the complete vector of ambiguities and, instead, finds a subset to maximize the availability of the solution. In this contribution, a new precision-aided PAR subset selection criteria based on the projection of the integer ambiguities into the positioning domain, using quaternion rotations, is proposed for GNSS-based attitude determination. Numerical simulations are provided to support the discussion and showcase the attitude determination performance, for both conventional Full Ambiguity Resolution (FAR) and precision-aided PAR approaches.

II. Preliminary on Quaternions for Attitude Estimation

Attitude determination is the process of finding the relative orientation between two orthogonal frames. While a plethora of attitude parametrizations exists, the use of (unit-norm) quaternions is widely extended. The main reasons behind the success of quaternions are: *a)* presenting a minimal state representation among non-singular attitude parametrizations; *b)* unconstrained estimators preserving the geometrical constraints can be easily derived by leveraging on Lie Theory. Unit quaternions are expressed as

$$\mathbf{q} \triangleq \begin{bmatrix} \cos(\theta/2) \\ \mathbf{u}\sin(\theta/2) \end{bmatrix} \in \mathcal{S}^3 \,, \tag{1}$$

with $\mathbf{u}$ an unit vector with the rotation axis and θ the rotation angle. Unit quaternions conform the manifold of 3D unit spheres $\mathcal{S}^3$ and a group under the quaternion multiplication. The rotation operator based on the use of unit quaternions is given by

$$r({}_{\mathcal{B}}\mathbf{v}) = \mathbf{q} \circ {}_{\mathcal{B}}\mathbf{v} \circ \mathbf{q}^* = \mathbf{R}_{\mathcal{B}}\mathbf{v}, \tag{2}$$

with $\circ$ the quaternion multiplication and $\mathbf{q}^*$ the inverse quaternion. Details on quaternion properties and a short introduction to Lie Theory can be consulted in [7], [8].

III. GNSS-Based Attitude Model

Similarly to RTK processing, the GNSS-based attitude model uses the double difference (DD) combination of observations to eliminate atmospheric products and eliminate carrier biases. In multi-antenna configurations, an antenna is considered *primary* with its position being the center of the body frame, while the remaining N antennas are denoted *secondaries*. Their positions are surveyed and accurately known within the platform frame, with the baseline vectors relating certain two antennas.

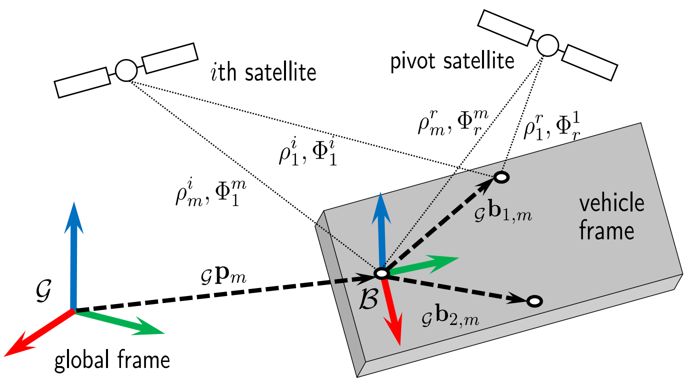

Fig. 1: Diagram for the antennas and satellites involved in the GNSS-based attitude model and diagram for the vehicle frame.

For instance, the vector that relates the positions of the primary and jth antennas is expressed as

$$ {}_{\mathcal{B}}\mathbf{b}_{j,m} = {}_{\mathcal{B}}\mathbf{p}_j - {}_{\mathcal{B}}\mathbf{p}_m, \tag{3} $$

where the left subscript denote the antenna under consideration, with m for the primary and $j = 1, \ldots, N$ for the secondary antennas, respectively. A total of $n+1$ satellites are assumed to be simultaneously tracked across all antennas. This notation is illustrated in Fig. 1, along with the GNSS code and carrier phase observations.

Let us define the code and carrier DD observation for the ith satellite and jth secondary antenna as follows

$$ \begin{aligned} DD\rho_{j,m}^i &= \rho_j^i - \rho_m^i - \left(\rho_j^r - \rho_m^r\right) \\ &= -\left(\mathbf{u}^i - \mathbf{u}^r\right)^\top \left(\mathbf{q} \circ {}_{\mathcal{G}}\mathbf{b}_{j,m} \circ \mathbf{q}^*\right) + \varepsilon_{j,m}^{i,r} \end{aligned} \tag{4} $$

$$ \begin{aligned} DD\Phi_{j,m}^i &= \Phi_j^i - \Phi_m^i - \left(\Phi_j^r - \Phi_m^r\right) \\ &= -\left(\mathbf{u}^i - \mathbf{u}^r\right)^\top \left(\mathbf{q} \circ {}_{\mathcal{G}}\mathbf{b}_{j,m} \circ \mathbf{q}^*\right) + \lambda a_j^i + \epsilon_{j,m}^{i,r} \end{aligned} \tag{5} $$

with $\mathbf{q} \circ {}_{\mathcal{G}}\mathbf{b}_{j,m} \circ \mathbf{q}^*$ the vehicle-to-global rotation operator for the baseline vector between the primary and jth secondary antenna, and $\mathbf{u}^i$ the line-of-sight vector to the ith satellite.

Let us denote the total number of DD observations as $M = n \cdot N$, and the complete vector of observations, $\mathbf{y} \in \mathbb{R}^{2M}$, as

$$ \mathbf{y} \triangleq \text{vec}\left(\mathbf{Y}\right), \quad \text{with } \mathbf{Y} = \left[\mathbf{y}_{1,m}, \ldots, \mathbf{y}_{N,m}\right], \tag{6} $$

where the observations for each $j-m$ pair of antennas is given by

$$ \mathbf{y}_{j,m}^\top = \left[\,[DD\Phi_{j,m}^1, \ldots, DD\Phi_{j,m}^n], [DD\rho_{j,m}^1, \ldots, DD\rho_{j,m}^n]\right]. \tag{7} $$

The GNSS attitude model constitutes a special case for the *mixed* estimation problem, in which integer and on-manifold parameters are unknown. The formal definition for the model and the associated estimation process is explained next.

A. *Estimation for the Attitude Mixed Model*

Let the attitude mixed model define the statistical distribution for code and carrier phase observations across multiple antennas on a vehicle frame such that

$$ \mathbf{y} \sim \mathcal{N}\left(\mathbf{A}\mathbf{a} + \mathbf{h}(\mathbf{q}), \boldsymbol{\Sigma}\right), \ \mathbf{a} \in \mathbb{Z}^M, \mathbf{q} \in \mathcal{S}^3 \tag{8} $$

with $\boldsymbol{\Sigma}$ the $2M \times 2M$ observations covariance matrix, $\mathbf{A}$ and $\mathbf{h}(\cdot)$ the design matrix and observation function as defined in [9, Ch. 3].

Estimating the unknowns in (8) leads to an optimization problem with mixed integer and on-manifold parameter estimation. From a maximum likelihood estimation (MLE) perspective, its computation follows a weighted least-squares (LS) formulation

$$ (\check{\mathbf{a}}, \check{\mathbf{q}}) = \arg \min_{(\mathbf{a},\mathbf{q}) \in \mathbb{Z}^M \times \mathcal{S}^3} \|\mathbf{y} - \mathbf{A}\mathbf{a} - \mathbf{h}(\mathbf{q})\|_{\boldsymbol{\Sigma}}^2, \tag{9} $$

where $\|\cdot\|_{\boldsymbol{\Sigma}}^2 = (\cdot)^\top \boldsymbol{\Sigma}^{-1} (\cdot)$ is a weighted norm. While an explicit solution for (9) is not known, the decomposition of the quadratic form into the sum of three LS adjustments is well-known [10] and expressed as follows

$$ \begin{aligned} &\min_{\mathbf{a} \in \mathbb{Z}^M, \mathbf{q} \in \mathcal{S}^3} \|\mathbf{y} - \mathbf{A}\mathbf{a} - \mathbf{h}(\mathbf{q})\|_{\boldsymbol{\Sigma}}^2 = \\ &\|\hat{\mathbf{e}}\|_{\boldsymbol{\Sigma}}^2 + \min_{\mathbf{a} \in \mathbb{Z}^M} \left(\|\hat{\mathbf{a}} - \mathbf{a}\|_{\mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{a}}}}^2 + \min_{\mathbf{q} \in \mathcal{S}^3} \|\hat{\mathbf{q}}(\mathbf{a}) - \mathbf{q}\|_{\mathbf{P}_{\hat{\mathbf{q}}(\mathbf{a})}}^2 \right) \end{aligned} \tag{10} $$

with $\|\hat{\mathbf{e}}\|_{\boldsymbol{\Sigma}}^2$ the norm of residuals over the auxiliary float estimates $\hat{\mathbf{a}}$, $\hat{\mathbf{q}}$, which can computed via the following minimization

$$ (\hat{\mathbf{a}}, \hat{\mathbf{q}}) = \arg \min_{\hat{\mathbf{a}} \in \mathbb{R}^M, \hat{\mathbf{q}} \in \mathcal{S}^3} \|\mathbf{y} - \mathbf{A}\hat{\mathbf{a}} - \mathbf{h}(\hat{\mathbf{q}})\|_{\boldsymbol{\Sigma}}^2, \tag{11} $$

$\hat{\mathbf{q}}(\mathbf{a})$ the float solution for the quaternion $\hat{\mathbf{q}}$ *conditioned* on $\mathbf{a}$ having $\mathbf{P}_{\hat{\mathbf{q}}(\mathbf{a})}$ as variance-covariance matrix; and with the adjustments being commonly denoted as *float*, IAR and fixed solution estimations.

The three consecutive estimation processes begin the with the float solution estimation, for which maintaining the geometrical constraints for the rotation (i.e., the unit norm of the quaternion) improves the overall performance. Then, the IAR consists on estimating the integer ambiguities based on the vector of real-valued ones. To do so, the mapping $S(\cdot) : \mathbb{R}^M \to \mathbb{Z}^M$ relates each float ambiguity estimate to an integer value:

$$ \mathcal{S}\left(\hat{\mathbf{a}}\right) = \sum_{\mathbf{a} \in \mathbb{Z}^M} \omega_a\left(\hat{\mathbf{a}}\right) \mathbf{a} + \left(1 - \sum_{\mathbf{a} \in \mathbb{Z}^M} \omega_a\left(\hat{\mathbf{a}}\right)\right) \hat{\mathbf{a}}, \tag{12} $$

where

$$ \omega_a\left(\hat{\mathbf{a}}\right) = \begin{cases} 1 & \text{if } \hat{\mathbf{a}} \in \Omega_{\mathbf{a}} \\ 0 & \text{otherwise.} \end{cases} \tag{13} $$

There are various alternatives to define the size or aperture of the pull-in regions [11]–[13]. This work considers the fixed-failure rate ratio test (FF-RT) [14], where the failure rate P_f is used as a tuning parameter.

Finally, the minimization on the right hand side of the brackets in (10) improves the vector of real-valued parameters $\hat{\mathbf{q}}$ upon the integer ambiguities $\check{\mathbf{a}}$, driving to a high precision attitude denoted as *fixed solution*. The mean and covariance for the fixed solution, $\check{\mathbf{q}}$, $\mathbf{P}_{\check{\mathbf{q}}\check{\mathbf{q}}}$ are based on the projection of the integer ambiguities into the quaternion domain, as

$$\check{\mathbf{q}} = \hat{\mathbf{q}} \ominus \mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{a}}}\mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{a}}}^{-1}\left(\hat{\mathbf{a}} - \check{\mathbf{a}}\right), \tag{14}$$

$$\mathbf{P}_{\check{\mathbf{q}}\check{\mathbf{q}}} = \mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{q}}} - \mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{a}}}\mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{a}}}^{-1}\mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{q}}}, \tag{15}$$

with the fixed solution inheriting its high precision from the carrier phase observables. Notice that the precision gain occurs only when estimated integer ambiguities coincide with the true ones, but this information is unknown in a real system. Alternatively, a fixed solution is considered only when the probability of a correct ambiguities fixing is sufficiently high (i.e., when the validity test is passed) [11], [15].

Otherwise, the complete set of integer estimates is disregarded, i.e. $\check{\mathbf{a}} = \hat{\mathbf{a}}$, and the fixed solution does not adjust the original float solution. PAR is a distinct alternative for finding the integer solution for only a subset of ambiguities. PAR methods are generally classified in two categories: model- and data-driven schemes [6], [16]. Model-driven (MD-PAR) methods base the subset selection $\mathcal{I}$ only on co-covariance matrix $\mathbf{P}_{\hat{\mathbf{a}}}$ information, while a data-driven (DD-PAR) approach integrate also the float ambiguities vector $\hat{\mathbf{a}}$. However, the effect of projecting the resolved integer ambiguities into the position domain as criteria for the subset selection called precision-aided PAR scheme has been addressed in [17], [18] giving good approaches in finding the integer solution. The precision-aided PAR (PD-PAR) for attitude determination is explained in the next Section.

IV. Precision-Driven PAR Scheme for Attitude Determination

Following the notation of [4] and [16], let $\mathcal{I}$ be the index for the subset of ambiguities to be fixed, such that

$$\mathcal{I} \subseteq \{1, \ldots, M\}, \quad \mathcal{I} \in \mathfrak{J}, \qquad \mathcal{I} \cap \bar{\mathcal{I}} = \emptyset, \quad \mathcal{I} \cup \bar{\mathcal{I}} = \{1, \ldots, M\} \tag{16}$$

where $\mathfrak{J}$ denotes the set of possible non-empty index combinations with cardinality $|\mathfrak{J}| = 2^M - 1$ and the complementary set $\bar{\mathcal{I}}$ indicates the ambiguities to remain real-valued. The real-to-integer mapping function now becomes $\mathcal{S} : \mathbb{R}^M \to \mathbb{Z}^{|\mathcal{I}|}$, and it is different among estimators. In general, the use of PAR leads to a suboptimal solution for the mixed problem in (9), since one intends at solving the alternative

$$\min_{\mathbf{a}_{\mathcal{I}} \in \mathbb{Z}^{\mathcal{I}}, \mathbf{a}_{\bar{\mathcal{I}}} \in \mathbb{R}^{|\bar{\mathcal{I}}|}, \mathbf{q} \in \mathbb{S}^3} \left\| \mathbf{y} - \mathbf{A}_{\mathcal{I}}\mathbf{a}_{\mathcal{I}} - \mathbf{A}_{\bar{\mathcal{I}}}\mathbf{a}_{\bar{\mathcal{I}}} - \mathbf{h}(\mathbf{q}) \right\|_{\mathbf{\Sigma}}^2, \tag{17}$$

where $\check{\mathbf{a}}_{\mathcal{I}}$, $\check{\mathbf{a}}_{\bar{\mathcal{I}}}$ and $\check{\mathbf{q}}$ are the arguments for (17), and with

$$\mathbf{A}_{\mathcal{I}} = \begin{bmatrix} \lambda_c \mathbf{I}_{|\mathcal{I}|} \\ \mathbf{0}_{|\mathcal{I}|,|\mathcal{I}|} \end{bmatrix}, \ \mathbf{A}_{\bar{\mathcal{I}}} = \begin{bmatrix} \lambda_c \mathbf{I}_{|\bar{\mathcal{I}}|} \\ \mathbf{0}_{|\bar{\mathcal{I}}|,|\bar{\mathcal{I}}|} \end{bmatrix}, \tag{18}$$

and only when $\bar{\mathcal{I}} = \emptyset$, (17) is equivalent to the original mixed estimation (9). While it appears illogical, aiming at solving a suboptimal problem, the use of PAR may improve the overall performance of an estimator for the mixed model by increasing the success rate for *some* ambiguities in contrast to *all* of them.

Although data- and model-driven schemes are widely used, they only take into account the ambiguities derived from the real-valued parameters estimation and the information brought by the covariance matrix of the ambiguities. However, precision-aided PAR (PD-PAR) fulfills the requirement of a minimal precision as selection criterion in the subset selection using the projection of the ambiguities into the fixed positioning domain. As the precision of the fix solution is conditioned on the quality of the float estimates and their associated co-covariance matrix, PD-PAR identifies the combination of ambiguities which grants a target precision requirement prior to the actual integer estimation.

Thus, one aims at finding a reduced number of ambiguities which guarantee certain target positioning precision criteria α for the fixed position solution, while retaining a sufficiently low failure rate P_{f_0}. Notice that the precision requirement α refers to the minimal precision criteria required by a particular application (e.g., automobile lane detection may require decimeter-level precision, while vessel mooring assistance might entail a precision of a few centimeters).

Thus, the PAR problem in (17) can be reformulated to be subject to a minimal precision criteria accuracy, as

$$\begin{aligned} &\min_{\mathbf{a}_{\mathcal{I}} \in \mathbb{Z}^{\mathcal{I}}, \mathbf{a}_{\bar{\mathcal{I}}} \in \mathbb{R}^{|\bar{\mathcal{I}}|}, \mathbf{q} \in \mathbb{S}^3} \left\| \mathbf{y} - \mathbf{A}_{\mathcal{I}}\mathbf{a}_{\mathcal{I}} - \mathbf{A}_{\bar{\mathcal{I}}}\mathbf{a}_{\bar{\mathcal{I}}} - \mathbf{h}(\mathbf{q}) \right\|_{\mathbf{\Sigma}}^2, \\ &\text{s.t. } \operatorname{tr}\left(\mathbf{P}_{\check{\mathbf{q}}\check{\mathbf{q}}}\right) \leq \alpha^2, \end{aligned} \tag{19}$$

where tr$(\cdot)$ denotes the trace operator. Unlike (14) and (15), the fixed solution for a PAR estimator is expressed in terms of the subset of ambiguities fixed, as

$$\check{\mathbf{q}} = \hat{\mathbf{q}} \ominus \mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{a}}_{\mathcal{I}}}\mathbf{P}_{\hat{\mathbf{a}}_{\mathcal{I}}}^{-1}\left(\hat{\mathbf{a}}_{\mathcal{I}} - \check{\mathbf{a}}_{\mathcal{I}}\right), \tag{20}$$

$$\mathbf{P}_{\check{\mathbf{q}}\check{\mathbf{q}}} = \mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{q}}} - \mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{a}}_{\mathcal{I}}}\mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{a}}_{\mathcal{I}}}^{-1}\mathbf{P}_{\hat{\mathbf{a}}_{\mathcal{I}}\hat{\mathbf{q}}}, \tag{21}$$

and, since $\mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{q}}}$ remains invariant with the subset choice, the selection can be realized so that

$$\operatorname{tr}\left(\mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{a}}_{\mathcal{I}}}\mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{a}}_{\mathcal{I}}}^{-1}\mathbf{P}_{\hat{\mathbf{a}}_{\mathcal{I}}\hat{\mathbf{q}}}\right) \geq \operatorname{tr}\ \left(\mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{q}}}\right) - \alpha^2, \tag{22}$$

so that one may omit performing integer estimation if the associated positioning precision does not match the target α. The procedure to operate PD-PAR consists on recursively finding the subset with best associated precision and whether a reliable integer solution exists (i.e., passing the validity test assures that the success rate is sufficiently high). If the position precision criteria α is not fulfilled, a fixed solution cannot be estimated for the subset $\mathcal{I}$. The subset $\mathcal{I}$ searching is based on (22) that follow from (15). Instead, if the precision is sufficient but a reliable solution is unavailable, the size of the subset reduces and the recursion is repeated.

Alg. 1 proposes a top-bottom (the number of ambiguities to integer-map decreases with the iterations) workflow for PD-PAR, with $\binom{M}{s} = M!/(s!(M-s)!)$ the binomial coefficient where n is the length vector of DD carrier phase and code observables and s is the number of discarded observations. Notice that the Z-transform is estimated for each subset size which greatly reduce the degree of decorrelation among ambiguities at the cost of a slightly superior computational complexity. Furthermore, whenever the satellite geometry is poor or the model is weak, one can rapidly disregards any integer estimation, provided that a potential fixed solution would not comply with a target positioning precision.

Notice that the computational complexity is dominated by the *subset listing* and *find best subset* operations in Alg. 1 with $\mathcal{O}(2^M + M^4)$ being the asymptotic time complexity of the algorithm. This can be substantially higher than current methods, however we would like to highlight that the additional computational complexity can be dealt with by the ever growing computational power of today's GNSS devices [19], [20], as we observed when running our experiments.

Algorithm 1: Precision-Driven PAR

Input : Float estimate: $\begin{bmatrix}\hat{\mathbf{a}}\\ \hat{\mathbf{q}}\end{bmatrix}, \begin{bmatrix}\mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{a}}} & \mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{q}}}\\ \mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{a}}} & \mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{q}}}\end{bmatrix}, P_{f_0}, \alpha$

Output: PD-PAR fixed solution: $\check{\mathbf{q}}, \check{\mathbf{a}}_{\mathcal{I}}$

Initialize $s = 0$.

while $s \geq M$ **do** *(iterate over subset size)*

- List subsets:
  $\mathcal{I}' \subseteq \{1, \ldots, M\},\ \mathcal{I}' \in \mathfrak{I}',\ |\mathfrak{I}'| = \binom{M}{M-s}$
- Find best subset:
  $\mathcal{I} = \arg\max_{\mathcal{I}'} \operatorname{tr}\left(\mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{a}}_{\mathcal{I}'}} \mathbf{P}^{-1}_{\hat{\mathbf{a}}\hat{\mathbf{a}}_{\mathcal{I}'}} \mathbf{P}_{\hat{\mathbf{a}}_{\mathcal{I}'}\hat{\mathbf{q}}}\right)$
- **if** $tr\left(\mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{a}}_{\mathcal{I}'}} \mathbf{P}^{-1}_{\hat{\mathbf{a}}\hat{\mathbf{a}}_{\mathcal{I}'}} \mathbf{P}_{\hat{\mathbf{a}}_{\mathcal{I}'}\hat{\mathbf{q}}}\right) < tr\left(\mathbf{P}_{\hat{\mathbf{q}}\hat{\mathbf{q}}}\right) - \alpha^2$ *(precision test not passed)* **then**
  - **return** $\check{\mathbf{a}}_{\mathcal{I}} = \hat{\mathbf{a}}_{\mathcal{I}}$ *(fixed solution unavailable)*
- **else**
  - Apply Z-transform and sorting $(\sigma_{\hat{z}_{n-s|I}} \leq \cdots \leq \sigma_{\hat{z}_{1|I}})$:
    $\hat{\mathbf{z}}_{\mathcal{I}} = \mathbf{Z}\hat{\mathbf{a}}_{\mathcal{I}},\ \mathbf{P}_{\hat{\mathbf{z}}\hat{\mathbf{z}}_{\mathcal{I}}} = \mathbf{Z}\mathbf{P}_{\hat{\mathbf{a}}\hat{\mathbf{a}}_{\mathcal{I}}}\mathbf{Z}^{\top}$.
  - Integer estimation: $\mathcal{S}(\hat{\mathbf{z}}_{\mathcal{I}})$
  - **if** $\mathcal{S}(\hat{\mathbf{z}}_{\mathcal{I}}) \in \mathbb{Z}^{|\mathcal{I}|}$ *(validity test passed)* **then**
    - **return** $\check{\mathbf{a}}_{\mathcal{I}} = \mathbf{Z}_{\mathcal{I}}^{-\top}\check{\mathbf{u}}_{\mathcal{I}},\ \check{\mathbf{u}}_{\mathcal{I}} = \mathcal{S}(\hat{\mathbf{z}}_{\mathcal{I}})$, *(subset integer solution)*
  - **else** Shrunk subset
    - $s = s + 1$

Fixed solution estimation via (20),(21)

V. Evaluation Results

The analysis and evaluation of the Attitude determination with two different ambiguity resolutions schemes, i.e., FAR and the precision-aided PAR, is presented in the sequel.

Two hours of GNSS data were used for the simulation setup at an IGS MGEX station POTS0 in Potsdam, Germany, on March 26th 2019 (DOY 085 12:00 - 14:00 UTC) with a data interval of 30 seconds. An instantaneous combined GPS (L1 + L2) and Galileo (E1+E5a) dual-frequency system was evaluated with a cut-off elevation angle of 10°. The failure rate was set to $P_f = 0.1\%$. Figure 2 illustrates the number of GPS and Galileo satellites along the experiment duration. The analysis of the proposed precision-aided PAR scheme was made for different baseline lengths, and implemented in a non-recursive (snapshot) LS-type *float* solution manner. The experimental results were accomplished with 10^4 Monte Carlo runs.

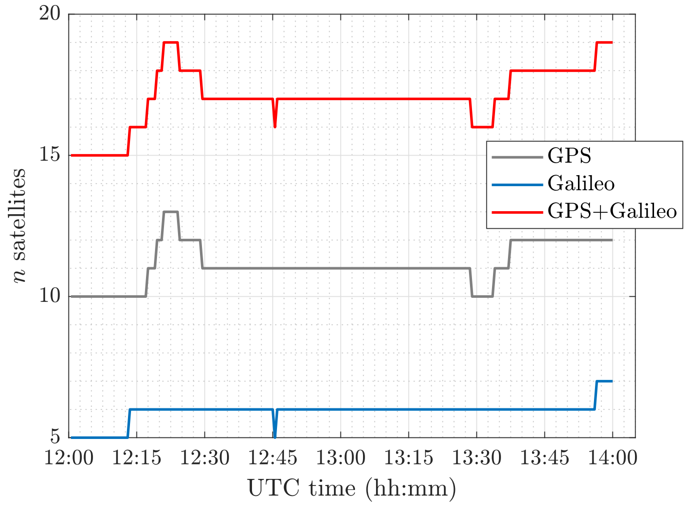

Fig. 2: Number of GPS (L1 + L2) and Galileo (E1+E5a) satellites for the simulated scenario.

Since the ionospheric delay is enough correlated in the base and rover station, it can be modeled as a zero-mean Gaussian random variable. However, when the baseline length increases over a few kilometers, the ionospheric delay must be taken into account. Thus, the ionospheric delay can be modeled as a noise with a distance dependent standard deviation $\sigma_L = 0.8\text{mm/km}$. Differential tropospheric delays are assumed to be zero.

The zenith-referenced (undifferenced) code and carrier phase standard deviations are listed in Table I. The ionospheric delays and the zenith-references code and carrier phase noises listed in Table I are scaled with the elevation dependent function $1/\sin(el)$.

TABLE I: Wavelengths and zenith-referenced code and carrier standard deviations for GPS and Galileo observations.

	GPS		Galileo	
	L1	L2	E1	E5a
λ (cm)	19.03	24.42	19.03	25.48
σ_c (cm)	37	28	35	28
σ_ϕ (mm)	2	2	2	2

The experimental setup was performed for a different number of satellites $n = \{8, 10, 12\}$ with a standard deviation for code observations $\sigma_c = 30\text{cm}$ which is considered a feature of a harsh environment. For every Monte Carlo run, the satellites were randomly discarded. Figure 3 illustrates the success ratio when the set of the integer ambiguities are correct estimated between the classical baseline-Tracking and quaternion-Tracking in function of the baseline length. It is noticeable that an attitude model does not depend of the baseline length since the problem is reduced to a mixed real-integer values parameter estimation. Hence, when we have a minimal of observations, the conventional baseline-Tracking success rate gives a poor performance below of 60% in comparative with quaternion-Tracking with a conventional FAR. The advantage in having a successful IAR is reflected when a PD-PAR method is used and the performance is improved for both-tracking methods. This accomplishment achieves 100% when the number of observations decreases assuring a successful IAR and by consequence an available attitude solution.

VI. Summary and Outlook

Attitude determination via GNSS carrier phase observations has been formulated as an Integer Ambiguity Resolution problem. This work proposes a novel PAR subset selection criterion based on the projection of the integer ambiguities into the positioning domain granting a target positioning precision requirement. Simulation results under realistic scenarios showed that a notorious improvement is reached for Quaternion-Tracking in comparative with the conventional Baseline-Tracking method. An assessment with PD-PAR for attitude attitude determination presents an achievement of 100% even when the number of observations decreases. As future work, an analysis in Harsh environments with Real-Time data is proposed.

VII. Acknowledgments

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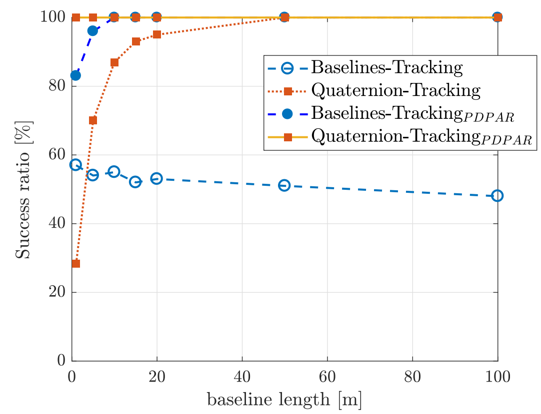
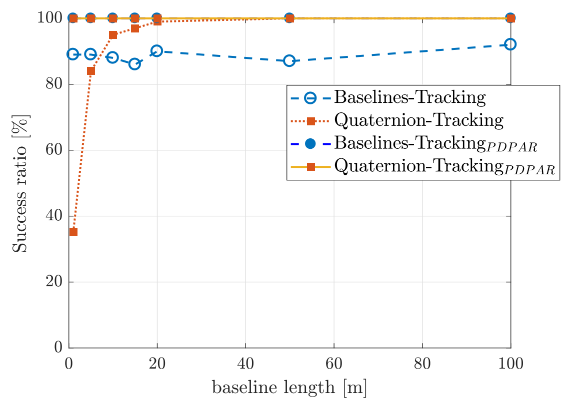
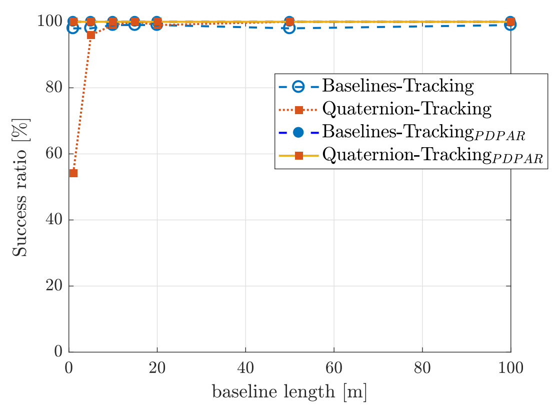


(a) 8 satellites



(b) 10 satellites



(c) 12 satellites

Fig. 3: Success ratio against baseline length for n=8, 10, 12 satellites in view, given a code noise of $\sigma_c = 30$ cm.

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