Robust pole assignment for descriptor systems

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Keywords: Descriptor systems, pole assignment, numerical methods.

Abstract

By using a generalized Sylvester equation based parametrization, three *minimum norm robust pole assignment* problems for descriptor systems are formulated as unconstrained minimization problems for suitably chosen cost functions. The derived explicit expressions of the gradients of the cost functions allow the efficient solution of the minimization problems by using powerful gradient search based minimization techniques. We also discuss how requirements for a particular Jordan structure of the closed-loop state matrix or for partial pole assignment can be accommodated with the proposed approach.

1 Introduction

Pole assignment techniques to modify the dynamic response of linear systems are among the most studied problems in modern control theory. The complete theoretical solution of this problem for standard systems has been followed by the development of many computational methods (see for example the collection of reprints in [13]). Sensitivity analysis of the pole assignment problem (see [9] and references therein) moves one step forward the understanding of difficulties and practical limitations associated with the usage of solution methods.

We address the solution of the pole assignment problem for the following descriptor system

$$E\lambda x(t) = Ax(t) + Bu(t), \tag{1}$$

where $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $\lambda x(t) = \dot{x}(t)$ for a continuous-time system and $\lambda x(t) = x(t+1)$ for a discrete-time system. We assume that $r = \mathrm{rank}\, E$, the pencil $A - \lambda E$ is regular, i.e., $\det(A - \lambda E) \not\equiv 0$, and the system (1) is controllable, i.e. $\mathrm{rank}\,(A - \lambda E, B) = n$ for $\lambda \in \mathbb{C}$ and $\mathrm{rank}[\, E \;\; B\,] = n$. In what follows we denote with $\Lambda(A, E)$ the set of generalized eigenvalues of the pair (A, E) and with $\Lambda_f(A, E)$ the subset of finite generalized eigenvalues.

Let $\Gamma_p = \{\lambda_1, \ldots, \lambda_p\}$ be a given symmetric set of p values in the extended complex plane (some values could be possibly infinite). We consider the following *descriptor eigenvalue assignment problems* (DEAPs):

1. DEAP1: Given Γ_n determine the proportional-derivative feedback pair (F, K) with $F, K \in \mathbb{R}^{m\times n}$ such that
$$\Lambda(A + BF, E + BK) = \Gamma_n. \tag{2}$$
2. DEAP2: Given Γ_n and $\gamma, \beta \in \mathbb{R}$ not both zero and $\gamma/\beta \notin \Lambda(A, E) \cup \Gamma_n$, determine the state feedback matrix $F \in \mathbb{R}^{m\times n}$ such that
$$\Lambda(A + \gamma BF, E + \beta BF) = \Gamma_n. \tag{3}$$
3. DEAP3: Given Γ_r with only finite values, determine the state feedback matrix $F \in \mathbb{R}^{m\times n}$ such that $A + BF - \lambda E$ is regular and
$$\Lambda_f(A + BF, E) = \Gamma_r.$$

Note that the condition of regularity of the closed-loop pencils is explicitly required only for DEAP3, because for both DEAP1 and DEAP2 the regularity is implicitly contained in (2) and (3).

The DEAP1 appears in solving coprime factorizations problem of rational matrices [16], where also a generalized Schur method is described to solve this problem. The DEAP2 is a particular case of DEAP1 and arises also in the context of solving coprime factorization problems [12]. This problem can be solved by reducing it to a standard problem in [12]. For particular choices of γ and β, the DEAP2 reduces to DEAPs with pure proportional feedback ($\beta = 0$) or pure derivative feedback ($\gamma = 0$). For $E = I$, $\gamma = 1$ and $\beta = 0$, we obtain the formulation for a *standard eigenvalue assignment problem* (SEAP). DEAP3 has been extensively studied in the literature (see e.g., [6] and cited references therein). The solution of DEAP3 involves basically the elimination of all impulsive modes by moving them to desired finite positions. This has also important applications in solving coprime and normalized coprime factorization problems [16, 17].

In the multi-input case each of the DEAPs has a non-unique solution. Therefore it is reasonable to exploit the non-uniqueness by imposing additional conditions. One aspect which is desirable from a practical point of view is to determine feedback matrices with small gains. Intuitively this must be advantageous since small feedback gains lead to smaller control signals, and thus to less energy consumption. Small gains are also beneficial to reduce noise amplification. A second aspect important in pole assignment is to achieve a small condition number for the eigenvector matrices of the closed-loop system pencil. This is the goal of *robust pole assignment* as formulated for both standard and descriptor systems [7, 2, 15, 6]. Both these aspects are decisive for the overall sensitivity of assigned eigenvalues, because, as was shown in the standard systems case [9], high feedback gains or high condition numbers lead to increased sensitivity of the closed-loop eigenvalues. It appears thus that the simultaneous minimization of the feedback norm and condition of eigenvector matrix is a desirable general goal for solving the DEAP.

In this paper we address the problem of determining the minimum norm state feedback which solves the DEAP and simultaneously minimizes the sensitivity of the closed-loop eigenvalues. The solutions of formulated DEAPs are based on Sylvester systems based parametrizations and are done by minimizing suitable cost functions expressing the requirements for low norm feedback and well conditioned eigenvectors. For the efficient solution of the minimization problems powerful unconstrained minimization methods based on gradient search techniques can be employed. For this purpose, we derived explicit expressions of the gradients of the cost functions. We discuss several functional and numerical features of the proposed approach, as generality, flexibility (e.g., to assign a desired eigenstructure or to perform a partial pole assignment), numerical stability, computational efficiency. Numerical examples illustrate some of the features of the proposed approach to solve DEAPs.

2 Parametrization of solutions

2.1 DEAP1

Our parametrization is based on a straightforward Sylvester equation based formulation. Let assume that F and K are matrices which solve the DEAP1. It follows, that there must exist invertible matrices X and Y such that

$$Y^{-1}(A+BF)X = \widetilde{A}, \quad Y^{-1}(E+BK)X = \widetilde{E}, \quad (4)$$

where the matrices $\widetilde{A}$ and $\widetilde{E}$ are such that $\Lambda(\widetilde{A},\widetilde{E}) = \Gamma_n$. If we define $G := FX$ and $L := KX$, then (4) can be rewritten as a Sylvester system of matrix equations

$$\begin{array}{rcl} AX - Y\widetilde{A} + BG &=& 0 \\ EX - Y\widetilde{E} + BL &=& 0 \end{array} \quad (5)$$

which must be satisfied by X and Y.

Now we can try to solve the DEAP1 assuming that $\widetilde{A}$ and $\widetilde{E}$ are chosen such that $\Lambda(\widetilde{A},\widetilde{E}) = \Gamma_n$, and G and L are given parameter matrices. To solve the DEAP1, we need to solve (5) for X and Y and, provided X and Y are invertible, we compute the feedback matrices as

$$F = GX^{-1}, \quad K = LX^{-1}. \quad (6)$$

To enforce the invertibility of X and Y, the matrices $\widetilde{A}$, $\widetilde{E}$, G and L must fulfill some conditions: 1) the pair $(\widetilde{A}-\lambda\widetilde{E}, G-\lambda L)$ is observable; 2) $\Lambda(A,E)\cap\Lambda(\widetilde{A},\widetilde{E}) = \emptyset$. These conditions together with the controllability of pair $(A-\lambda E, B)$ ensure that X and Y satisfying (5) are generically invertible. Note that by choosing the pair $(\widetilde{A},\widetilde{E})$ in a Weierstrass canonical form then the resulting X and Y play the role of the closed-loop eigenvector matrices for the closed-loop system pair $(A+BF, E+BK)$.

2.2 DEAP2

For the solution of the DEAP2 an entirely similar parametrization can be employed as for DEAP1. Assuming $\widetilde{A}$ and $\widetilde{E}$ are chosen such that $\Lambda(\widetilde{A},\widetilde{E}) = \Gamma_n$, and G is a given parameter matrix, the feedback F is computed as

$$F = GX^{-1} \quad (7)$$

and X satifies the Sylvester system

$$\begin{array}{rcl} AX - Y\widetilde{A} + \gamma BG &=& 0 \\ EX - Y\widetilde{E} + \beta BG &=& 0 \end{array} \quad (8)$$

The controllability of the pair $(A-\lambda E, B)$ and the choice of $\widetilde{A}$, $\widetilde{E}$ and G such that: 1) the pair $(\widetilde{A}-\lambda\widetilde{E}, (\gamma-\lambda\beta)G)$ is observable; 2) $\Lambda(A,E)\cap\Lambda(\widetilde{A},\widetilde{E}) = \emptyset$, ensure that X satisfying (8) is generically invertible.

We also present an alternative parametrization which relies on converting the DEAP2 by means of a conformal mapping (see also [12]) to a DEAP with E nonsingular and Γ_n with only finite values. By straightforward manipulation we get the equivalent Sylvester system

$$\begin{array}{rcl} (\gamma A+\beta E)X - Y(\gamma\widetilde{A}+\beta\widetilde{E}) + (\gamma^2+\beta^2)BG &=& 0 \\ (\beta A-\gamma E)X - Y(\beta\widetilde{A}-\gamma\widetilde{E}) &=& 0 \end{array}$$

By assumption γ/β is not an eigenvalue of the pair (A,E) and also not an eigenvalue to be assigned. Thus, by solving the second equation for Y and replacing Y in the first one we obtain

$$A_zX - E_zX\widetilde{A}_z + B_zG = 0 \quad (9)$$

where

$$\begin{array}{rcl} A_z &:=& \gamma A+\beta E, \\ E_z &:=& \beta A-\gamma E, \\ B_z &:=& (\gamma^2+\beta^2)B \\ \widetilde{A}_z &:=& (\beta\widetilde{A}-\gamma\widetilde{E})^{-1}(\gamma\widetilde{A}+\beta\widetilde{E}) \end{array}$$

Observe that computing F as in (7) with X satisfying (9) is in fact a parametrization for a standard problem with E_z

nonsingular and $\Gamma_{z,n} := \Lambda(\widetilde{A}_z)$ having only finite values. Note that $\widetilde{A}_z$ results by applying the conformal mapping $\lambda = (\gamma z + \beta)/(\beta z - \gamma)$ to A. However, $\widetilde{A}_z$ can be also directly formed after applying the conformal mapping to Γ_n to get $\Gamma_{z,n}$.

With the transformed data we have reduced the original DEAP2 to the computation of F such that

$$\Lambda(A_z + B_z F, E_z) = \Gamma_{z,n}, \tag{10}$$

with E_z invertible. From the assumptions on the system matrices and on γ and β, we get

$$\text{rank}\,[\, A_z - zE_z \;\; B_z \,] = n, \quad \forall z \in \mathbb{C}.$$

Thus (10) is a SEAP for the controllable pair $(E_z^{-1}A_z, E_z^{-1}B_z)$ and the set $\Gamma_{z,n}$, and has always a solution F which is the solution of the original DEAP2 as well. Note that if $\widetilde{A}_z$ is in a Jordan form then X plays the role of the closed-loop eigenvector matrix for the standard system pair $(E_z^{-1}A_z, E_z^{-1}B_z)$.

The equation (9) can be rewritten as a Sylvester system of matrix equations

$$\begin{array}{rcl} A_z X - Y\widetilde{A}_z + B_z G &=& 0 \\ E_z X - Y &=& 0 \end{array} \tag{11}$$

Note that X is the same matrix in both (8) and (11), but Y in (8) is different from that in (11).

2.3 DEAP3

For the parametrization of this problem we choose the matrices $\widetilde{A}, \widetilde{E} \in \mathbb{R}^{n,n}$ in the particular forms

$$\widetilde{A} = \left[\begin{array}{cc} \widetilde{A}_r & 0 \\ 0 & I_{n-r} \end{array}\right], \quad \widetilde{E} = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array}\right]. \tag{12}$$

By using the parametrization approach for DEAP1 with $L = 0$, we get from (5)

$$\begin{array}{rcl} AX - Y\widetilde{A} + BG &=& 0, \\ EX - Y\widetilde{E} &=& 0. \end{array} \tag{13}$$

Partition X, Y and the given G columnwise in accordance with the structure of $\widetilde{A}$ in the form

$$X = [\, X_1 \; X_2 \,], \quad Y = [\, Y_1 \; Y_2 \,], \quad G = [\, G_1 \; G_2 \,]. \tag{14}$$

Then we obtain from (13) the system of matrix equations

$$\begin{array}{rcl} AX_1 - Y_1\widetilde{A}_r + BG_1 &=& 0 \\ EX_1 - Y_1 &=& 0 \\ AX_2 - Y_2 + BG_2 &=& 0 \\ EX_2 &=& 0 \end{array} \tag{15}$$

If we take X_2 as an orthogonal basis matrix for the nullspace of E, then Y_2 results accordingly from the third equation of (15). Thus we have to solve only the first two of the above equations, or alternatively the generalized Sylvester equation for X_1

$$AX_1 - EX_1\widetilde{A}_r + BG_1 = 0$$

which results after replacing $Y_1 = EX_1$ in the first equation. Note that if $\widetilde{A}_r$ is in a Jordan form then X_1 plays the role of the closed-loop eigenvector matrix corresponding to the assigned finite eigenvalues.

3 Solution of robust DEAPs

Consider first the solution of DEAP1. If the pair $(\widetilde{A}, \widetilde{E})$ is in a Weierstrass form then X and Y play the roles of the eigenvector matrices for the closed-loop system pair $(A+BF, E+BK)$. In light of the sensitivity results in [9] for standard systems it meaningful to exploit the non-uniqueness of the DEAP1 for multi-input systems by minimizing additionally the sensitivity of the closed-loop eigenvalues and the norm of the feedback matrix. This leads to a *minimum norm robust* DEAP for which we propose a solution method combining unconstrained optimization techniques with the parametric Sylvester equation based approach.

As a measure of the sensitivity of closed-loop eigenvalues, we use the condition numbers $\kappa_F(X)$ and $\kappa_F(Y)$ of X and Y with respect to the Frobenius norm. For computational convenience, instead of minimizing $\kappa_F(X) := \|X\|_F\|X^{-1}\|_F$, the minimization of the sum $\|X\|_F^2 + \|X^{-1}\|_F^2$ can be alternatively performed, since the two optimization problems are mathematically equivalent [2]. Thus, for the simultaneous minimization of the norm of the state feedback matrices F and K and of the two condition numbers $\kappa_F(X)$ and $\kappa_F(Y)$ we can use the following performance index

$$\begin{array}{rcl} J &=& \dfrac{\alpha}{2}\left(\|X\|_F^2 + \|X^{-1}\|_F^2 + \|Y\|_F^2 + \|Y^{-1}\|_F^2\right) \\ && + \dfrac{1-\alpha}{2}\left(\|F\|_F^2 + \|K\|_F^2\right), \end{array} \tag{16}$$

where $0 \le \alpha \le 1$ is a weighting factor. For $\alpha = 0$ J defines a pure norm minimization problem, while for $\alpha = 1$ we get a pure robust DEAP. Intermediary values of α lead to a combination of both aspects. With a similar interpretation of matrices X and Y the same criterion J with $K = 0$ can be employed to solve the robust DEAP2 or robust DEAP3.

The main advantage of the Sylvester equation based parametrization is that it allows a straightforward derivation of analytic expressions of gradients of the performance criterion J with respect to the free parameters G and L. We have the following result for the DEAP1:

Proposition 1 *Let (F, K) be the pair of state feedback matrices computed as in (6), assigning the desired eigenvalues Γ_n for given $(\widetilde{A}, \widetilde{E})$ and (G, L). Then, the gradients of J with respect to G and L are given by*

$$\begin{array}{rcl} \nabla_G J &=& (1-\alpha)FX^{-T} + B^T U \\ \nabla_L J &=& (1-\alpha)KX^{-T} + B^T V \end{array} \tag{17}$$

where U and V satisfy the Sylvester system

$$\begin{array}{rcl} A^TU + E^TV &=& S \\ U\widetilde{A}^T + V\widetilde{E}^T &=& -T \end{array} \tag{18}$$

for

$$\begin{array}{rcl} S &=& (1-\alpha)(F^TF + K^TK)X^{-T} \\ && +\alpha(-X + X^{-T}X^{-1}X^{-T}) \\ T &=& \alpha(-Y + Y^{-T}Y^{-1}Y^{-T}). \end{array}$$

With obvious replacements we obtain immediately the analogous result for DEAP2.

Proposition 2 *Let F be the state feedback matrix computed as in (7), assigning the desired eigenvalues Γ_n for given (γ, β) and $(\widetilde{A}, \widetilde{E})$. Then, the gradient of J with respect to G is given by*

$$\nabla_G J = (1-\alpha)FX^{-T} + B^T(\gamma U + \beta V) \tag{19}$$

where U and V satisfy the Sylvester system

$$\begin{array}{rcl} A^TU + E^TV &=& S \\ U\widetilde{A}^T + V\widetilde{E}^T &=& -T \end{array} \tag{20}$$

for

$$\begin{array}{rcl} S &=& (1-\alpha)\,F^TFX^{-T} + \alpha\,(-X + X^{-T}X^{-1}X^{-T}) \\ T &=& \alpha\,(-Y + Y^{-T}Y^{-1}Y^{-T}). \end{array}$$

For the alternative formulation of the DEAP2 we have the following result:

Proposition 3 *Let F be the state feedback matrix computed as in (7), assigning the desired eigenvalues $\Gamma_{z,n}$ for given $\widetilde{A}_z$ and G. Then, the gradient of J with respect to G is given by*

$$\nabla_G J = (1-\alpha)FX^{-T} + B_z^TU^T \tag{21}$$

where U satisfies the Sylvester system

$$\begin{array}{rcl} A_z^TU + E_z^TV &=& S \\ U\widetilde{A}_z^T + V &=& -T \end{array} \tag{22}$$

with S and T as in Proposition 2.

The solution of robust DEAP3 has been considered in [6], where only the minimization of the condition number of X has been pursued. Alternatively, the condition numbers of both X and Y can be minimized simultaneously. For the DEAP3 we have the following result:

Proposition 4 *For X satisfying (13), let $F = GX^{-1}$ be the state feedback matrix assigning the desired finite eigenvalues Γ_r for given $(\widetilde{A}, \widetilde{E})$ of the form (12) and G. Let G be partitioned columnwise as $G = [\,G_1\ G_2\,]$ in accordance with $\widetilde{A}$ in (12). Then, the gradients of J with respect to G_1 and G_2 are given by*

$$\begin{array}{rcl} \nabla_{G_1} J &=& (1-\alpha)FX^{-T}\begin{bmatrix} I_r \\ 0 \end{bmatrix} + B^TU \\ \nabla_{G_2} J &=& (1-\alpha)FX^{-T}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} \\ && +\alpha\,B^T(Y - Y^{-T}Y^{-1}Y^{-T})\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} \end{array} \tag{23}$$

where U satisfies the Sylvester system

$$\begin{array}{rcl} A^TU + E^TV &=& S\begin{bmatrix} I_r \\ 0 \end{bmatrix} \\ U\widetilde{A}_r^T + V &=& T\begin{bmatrix} I_r \\ 0 \end{bmatrix} \end{array} \tag{24}$$

with S and T as in Proposition 2.

For each of DEAPs, each function and gradient evaluation involves the solution of two Sylvester systems (e.g., (5) and its dual (18) for DEAP1). An efficient algorithm to solve these equations is available [5]. In the next section we describe a transformation based approach by which gradient computations can be substantially speeded up.

Having explicit analytical expressions for the function and its gradient it is easy to employ any gradient based technique to minimize J. However, since the dimension of the minimization problem nm could be potentially large, a particularly well suited class of methods to solve our problems is the class of unconstrained descent methods, as for instance, the limited memory BFGS method [8] used in conjunction with a line search procedure with guaranteed decrease as that described in [11]. The guaranteed decrease feature of these methods ensures that for $\alpha > 0$ the condition numbers $\kappa_F(X)$ and $\kappa_F(Y)$ progressively decrease and thus the solutions X and Y of (5), (11) or (13) remain invertible at each iteration once invertible solutions have been determined at the first iteration.

A word of caution is necessary when using gradient techniques to solve DEAPs by minimizing J. Since the minimization problems has possibly many local minima, it is likely that the computed solution is a local minimum. By solving the problem repeatedly with different initializations, we can choose that solution which produces the lowest value of the cost function. Note however, that in most of cases the global minimum leads to condition numbers of transformation matrices X and Y which have the same order of magnitude as those corresponding to any of local minima. Thus there is practically no difference for solving a robust eigenvalue assignment problem if the global minimum or one of local minima is employed to compute the feedback.

4 Algorithmic features

A satisfactory eigenvalue assignment algorithm must fulfill several functional and numerical requirements to serve as basis for a numerically robust software implementation. In what follows we discuss several algorithmic features of the Sylvester equation approach and we point out how apparent limitations of this method can be overcome.

4.1 Functional features

A satisfactory computational algorithm must be general and flexible, and must be able to exploit all structural aspects of the underlying problem. We will examine these aspects in

detail in case of the eigenvalue assignment method based on the Sylvester equation approach.

Generality

Generality means that an eigenvalue assignment algorithm is able to assign an arbitrary set of eigenvalues and ideally, it can also assign a desired eigenstructure for the closed-loop system. Although the first requirement seems to be trivial, even well-known methods implemented in commercial software are not able to fulfill this requirement. For example, the robust pole assignment method of [7] can not assign poles with multiplicities greater than rank of B and the improved version of this approach has the same limitation [15]. The Sylvester approach has no such limitations, although for a complete generality two aspects must be additionally addressed: the assignment of a given eigenstructure for the closed-loop eigenvalues and the assignment of eigenvalues which possibly coincide with those of the original system.

An arbitrary set of eigenvalues can be assigned with the Sylvester equation based approach by suitably choosing the matrix pair $(\widetilde{A}, \widetilde{E})$ used in DEAP1 and DEAP1 or the matrices $\widetilde{A}_z$ and $\widetilde{A}_r$ used in the conformally transformed DEAP2 and DEAP3, respectively. For example, assume that Γ_n contains p distinct eigenvalues $\lambda_1, \ldots, \lambda_p$, and each eigenvalue λ_i has multiplicity k_i. Assume further that the last eigenvalue $\lambda_p = \infty$. For the DEAP1 and DEAP2 we can choose the pair $(\widetilde{A}, \widetilde{E})$ in a Weierstrass canonical form

$$\widetilde{A}-\lambda\widetilde{E} = \begin{bmatrix} J_{k_1}(\lambda_1) - \lambda I & & 0 & & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & & J_{k_{p-1}}(\lambda_{p-1}) - \lambda I & & 0 \\ 0 & & 0 & & I - \lambda J_{k_p}(0) \end{bmatrix}$$

where $J_k(\lambda)$ denotes a Jordan block for the eigenvalue λ of order k. For a complex eigenvalue λ_i belonging to a multiple pair of complex conjugated eigenvalues $(\lambda_i, \overline{\lambda}_i)$, a $2k_i \times 2k_i$ real Jordan block can be used instead of two $k_i \times k_i$ complex Jordan blocks [3, pag. 365]. If rank $B = m > 1$, better conditioned transformation matrices X and Y can be obtained by employing several Jordan blocks of lower dimensions for each multiple eigenvalue. Thus for each λ_i of multiplicity k_i, up to m Jordan blocks with dimensions at most $\left[\frac{k_i}{m}\right] + 1$ can be used, where $[\cdot]$ denotes the integer part. For the transformed DEAP2 and the DEAP3 the matrices $\widetilde{A}_z$ and $\widetilde{A}_r$ can be chosen with a similar Jordan structure.

Although the Sylvester equation based approach can be used for eigenstructure assignment, still there are some subtle limitations with respect to the admissible closed-loop eigenstructure. This can be seen even in the case of standard systems ($E = I$), where the eigenstructure assignment problem can be equivalently formulated as the assignment of a set of invariant polynomials ψ_1, ψ_2, ..., ψ_n for the closed loop state matrix $A + BF$. Let $q_1 \leq q_2 \leq \ldots \leq q_m$ be the controllability indices of the controllable pair (A, B). Then the eigenstructure assignment problem has a solution [14] if and only if the following set of inequalities is satisfied

$$\sum_{i=1}^{j} q_i \geq \sum_{i=1}^{j} \deg \psi_i, \quad j = 1, \ldots, m \tag{25}$$

where (25) holds with equality when $j = m$. Thus, for the assignment of a desired Jordan structure for the closed-loop state matrix $A + BF$ the choice of corresponding $\widetilde{A}$ must reflect this structure in accordance with the conditions (25).

The second aspect of overlapping open-loop and closed-loop eigenvalues can be easily solved with the help of a preliminary state-feedback. In case of DEAP1 we can always use as a *practical* solution two randomly generated matrices F_0 and K_0 and solve the DEAP1 for the modified pair $(A + BF_0 - \lambda(E + BK_0), B)$. If (F_1, K_1) is the solution of this problem, then $(F, K) = (F_0 + F_1, K_0 + K_1)$ solves the original problem. A similar approach can be used also for the DEAP2 and DEAP3. An alternative, numerically more robust solution to this aspect is discussed in the next paragraph.

The use of a preliminary feedback can accommodate also the case of non-regular pairs (A, E). For a controllable pair $(A - \lambda E, B)$ with $\det(A - \lambda E) \equiv 0$, there always exists a feedback pair (F_0, K_0) such that the modified pair $(A + BF_0, E + BK_0)$ is regular. For *practical* purposes, randomly generated preliminary feedback matrices F_0 and K_0 can be used. This approach is applicable for DEAP1 and DEAP3, but not for DEAP2 because the regularity of pair (A, B) is a necessary condition for the existence of a solution.

Flexibility

One apparent limitation of the Sylvester equation based approach is the need that the closed-loop and open-loop spectra do not overlap. This condition guarantees the existence of a unique solution to the generalized Sylvester equations (5), (11) and (13) and is thus necessary when using Sylvester equation solvers as those proposed in [4]. Although technical, this condition prevents the Sylvester equation based approach to perform a *partial eigenvalue assignment*, i.e., to keep unmodified some of the open-loop eigenvalues. Since the partial eigenvalue assignment is a very useful feature, especially in the case of stabilizing high order systems, we show how this feature can be accommodated within the Sylvester equation based approach and thus substantially increasing its flexibility.

It is easy to see that the performance index J is invariant to an orthogonal system similarity transformation, that is, if F and K are the optimal feedback matrices for the descriptor pair $(A - \lambda E, B)$ then $\widehat{F} = FZ$ and $\widehat{K} = KZ$ are the optimal feedback matrices for the transformed pair $(\widehat{A} - \lambda\widehat{E}, \widehat{B}) := (Q^T AZ - \lambda Q^T EZ, Q^T B)$, where Q and Z are orthogonal matrices. Thus, if we want to keep unmodified the generalized eigenvalues of pair (A, E) lying in a "good" region $\mathbb{C}_g$ of $\mathbb{C}$ and to modify only those lying in its complement $\mathbb{C}_b = \mathbb{C} \setminus \mathbb{C}_g$ (the "bad" region), then we can first reduce (A, E) to an ordered *generalized real Schur form*

(GRSF) to obtain the triple

$$\begin{aligned} Q^T E Z &= \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \\ Q^T A Z &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \\ Q^T B &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \end{aligned} \tag{26}$$

where $\Lambda(A_{11}, E_{11}) \subset \mathbb{C}_g$ and $\Lambda(A_{22}, E_{22}) \subset \mathbb{C}_b$. With this separation, we can perform a partial pole assignment by solving for the optimal solution F_2 and K_2 the DEAP1 for the reduced descriptor pair $(A_{22} - \lambda E_{22}, B_2)$. The overall optimal feedback matrices result as $F = [\ 0 \quad F_2\ ]Z^T$ and $K = [\ 0 \quad K_2\ ]Z^T$. A similar approach is possible for the DEAP2 and DEAP3.

Structure exploitation

The Sylvester equation approach in conjunction with the optimization based search for a minimum norm and well-conditioned feedback exploits the intrinsic freedom of the multi-input DEAP to address additional requirements, as for example the conditioning aspect of the problem. Note that most of pole assignment algorithms do not exploit this structural feature of the problem and even algorithms for robust pole assignment address only partially this aspect by ignoring norm minimization. Moreover, these methods have also restrictions with respect to the allocation of the closed-loop eigenstructure. In a larger context, the Sylvester equation based approach provides a unified framework to solve various eigenvalue assignment problems for standard, descriptor and even periodic systems [18]. In light of discussions on generality and flexibility, this approach has the potential to become the standard way to solve all classes of eigenvalue assignment problems.

4.2 Numerical features

We focus on discussing numerical properties like the numerical stability and computational efficiency of the Sylvester equation based eigenvalue assignment algorithms, and we address shortly the implementation aspects of this approach in robust numerical software.

Numerical stability

For DEAP1 the computation of the optimal solution F and K for the computed optimal parameter matrices G and L involves the solution of two systems of linear equations: the Sylvester system (5) to compute X and Y, and the linear system $\begin{bmatrix} F \\ K \end{bmatrix} X = \begin{bmatrix} G \\ L \end{bmatrix}$ to compute the feedback matrices F and K. Thus the Sylvester equation based approach can be considered to be practically numerically stable. This is also true for solving DEAP2 and DEAP3.

Concerning the accuracy of the results, in a robust pole assignment problem it is expected that the optimal X is reasonably well-conditioned, thus the last computational step is guaranteed very accurate. The main source of errors is the solution of the Sylvester system, and thus the separation of spectra of the pairs (A, E) and $(\widetilde{A}, \widetilde{E})$ is the essential factor here. However, a good separation can be always achieved by an initial eigenvalue shifting with a preliminary feedback (see also subsection 4.1). Thus, for most practical problems we can expect that the computed results corresponding to an optimal solution are very accurate.

Efficiency

The overall efficiency of the algorithms heavily depends on the costs of function and gradient evaluations. Each function and gradient evaluation involves the solution of two generalized Sylvester equations sharing the same coefficient matrices (e.g., (5) and (18)). The standard procedure to solve these equations is the well-known generalized Schur method [5] implemented in [4]. This approach can be efficiently employed in our case provided the pair (A, E) is reduced first to a GRSF using an orthogonal similarity transformation and assuming further that in the DEAP1 or DEAP2 the pair $(\widetilde{A}, \widetilde{E})$ is in a Weierstrass form (a particular GRSF) or that in the transformed DEAP2 and DEAP3 the matrices $\widetilde{A}_z$ or $\widetilde{A}_r$ are in Jordan forms, respectively. The reduction of (A, E), performed only once, requires about $25n^3$ operations and can be seamlessly combined with the reordering of the GRSF to accommodate with the partial pole assignment requirement. The solution of the minimization problem for DEAP1 can be performed to obtain the optimal solutions $\widehat{F}$ and $\widehat{K}$ for the transformed descriptor pair $(\widehat{A} - \lambda\widehat{E}, \widehat{B}) = (Q^T A Z - \lambda Q^T E Z, Q^T B)$ with the pair $(\widehat{A}, \widehat{E})$ in GRSF and $(\widetilde{A}, \widetilde{E})$ in Weierstrass form. The solution of the original DEAP1 results as $F = \widehat{F} Z^T$ and $K = \widehat{K} Z^T$. For the transformed problem, the function and gradient evaluations can be performed very efficiently since now we have to solve only reduced Sylvester equations with the coefficient matrices in GRSF. This involves about $2n^3$ operations for the solution of each Sylvester system by using the algorithm of [5]. Thus the overall cost to evaluate the function and its gradient is about $10n^3$ operations, from which $6n^3$ operations amounts to form the free terms S and T in (18).

Implementation aspects

The Sylvester equation based approach is very simple to implement. For a FORTRAN implementation, all necessary software to perform the linear algebra computations is available in LAPACK 3.0 [1]. Here routines to compute the GRSF of a matrix pair, to solve Sylvester and dual Sylvester systems, as well as systems of linear equations are provided. For optimization, efficient unconstrained minimization routines are available in MINPACK-2 (the successor of MINPACK-1 [10]), offering a convenient reverse communication interface

which allows an easy implementation of function and gradient computations.

For testing purposes, prototype MATLAB functions have been implemented by the author to solve all three DEAPs. The corresponding functions rely on an efficient *mex*-function to solve Sylvester systems developed within the NICONET project[1]. For optimization, the `fminunc` unconstrained minimization function available in the Optimization Toolbox 2.0 of MATLAB has been employed.

5 Numerical examples

Consider the system from [6] with the matrices

$$A = \begin{bmatrix} 0 & 1.1 & 0 & 0 & 0 \\ 0 & 0 & 1.56 & 0 & 0 \\ 1.23 & 0 & 0 & 1.98 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.01 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 1.55 & 0 & 0 \\ 0 & 1.07 & 0 \\ 0 & 0 & -1.11 \\ 0 & -2.5 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 1.72 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.82 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This system is not regular since $\det(A - \lambda E) \equiv 0$ but can be made regular with a preliminary state feedback. We assigned the closed-loop eigenvalue set $\Gamma_5 = \{-0.5, -1, -2, \infty, \infty\}$ using all three Sylvester equation based approaches for DEAP1, DEAP2 and DEAP3. To solve DEAP2 a necessary condition is the regularity of the pencil $A - \lambda E$. This is while, in this case we used a modified system. The examples illustrate not only the ability of the new approach to compute well-conditioned (sometimes almost orthogonal) eigenvector matrices, but also the ability of this approach to solve the DEAP1 and DEAP3 even for non-regular systems or to solve DEAP2 in case when infinite eigenvalues are present in both the open-loop and closed-loop systems.

For the solution of DEAP1 we chose

$$\widetilde{A} = \mathrm{diag}(-0.5, -1, -2, 1, 1), \quad \widetilde{E} = \mathrm{diag}(1, 1, 1, 0, 0)$$

and solved the minimization problem for J for several values of α. For the resulting optimal solutions we computed in each case the 2-norm condition numbers $\kappa_2(X)$ and $\kappa_2(X)$, and the 2-norm of the compound feedback matrix $[\,F\ K\,]$. The results are given in Table 1:

It is easy to observe that for decreasing values of α the norms of feedback matrices decrease, but the condition numbers of eigenvector matrices increase. For pure norm minimization, a very small norm has been achieved but with a very ill-conditioned transformation matrix Y. Each solution for α in the range $[10^{-2}, 1]$ is practically acceptable.

[1] see `http://www.win.tue.nl/niconet/niconet.html`

α	$\|[\,F\ K\,]\|_2$	$\kappa_2(X)$	$\kappa_2(Y)$
1	1.35	3.75	1.57
0.1	0.58	3.97	3.18
0.01	0.33	4.25	6.52
0.001	0.21	5.06	16.98
0.0001	0.13	9.86	57.07
0	0.014	445	$1.88{\cdot}10^7$

Table 1: Results for DEAP1

For reference purpose we give the feedback matrices computed with $\alpha = 1$ for the pure robust DEAP1:

$$F = \begin{bmatrix} 0.0713 & -0.2135 & -0.1447 & 0.6371 & -0.1770 \\ -0.0708 & 0.1770 & 0.4130 & 0.3929 & 0.0373 \\ -0.2483 & -0.2858 & 0.0012 & -0.3999 & 0.9293 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.0705 & 0.1758 & -0.5965 & -0.2388 & -0.0443 \\ 0.1425 & 0.0033 & -0.0111 & -0.4988 & 0.3255 \\ 0.3163 & 0.0623 & -0.2114 & 0.4182 & -0.1678 \end{bmatrix}$$

The robustness of the solution can be easily checked by computing the eigenvalues of the pair $(A+BF, E+BK)$ for the matrices F and K truncated to the displayed four digits. The resulting closed-loop eigenvalues are:

$$\{-1.999997, -0.999998, -0.50004, \infty, \infty\}$$

and thus are accurate to 5 decimal digits.

To solve the DEAP2 the pencil $A-\lambda E$ must be regular. To ensure this, we use instead E a new matrix $\widehat{E}$ which is equal to E excepting the element $(4,4)$ which is set to 1. The modified pair $(A, \widehat{E})$ is regular and its generalized eigenvalues are $\{0, 0, -1.5, \infty, \infty\}$. To assign the same eigenvalues as in the previous example, we see that the infinite eigenvalues coincide. An initial feedback has been used to perturb the infinite eigenvalues to allow the application of the Sylvester equation based approach. We chose $\gamma = 1$ and $\beta = 1$ and we used the parametrization based on (8) with the same matrix pair $(\widetilde{A}, \widetilde{E})$ as for DEAP1. For the same values of α as in Table 1 we obtained the results in Table 2.

α	$\|F\|_2$	$\kappa_2(X)$	$\kappa_2(Y)$
1	0.57	4.21	6.59
0.1	0.41	4.00	6.70
0.01	0.31	4.87	7.21
0.001	0.22	9.35	9.83
0.0001	0.24	10.51	10.25
0	0.24	10.51	10.25

Table 2: Results for DEAP2

The feedback matrix computed with $\alpha = 1$ for the pure robust DEAP2 is

$$F = \begin{bmatrix} -0.030119 & -0.01914 & -0.11327 & 0.41749 & 0.13902 \\ 0.034855 & 0.02796 & 0.16545 & 0.06424 & 0.20172 \\ 0.038725 & -0.00246 & -0.01460 & 0.35707 & -0.00433 \end{bmatrix}$$

To check the robustness of the solution we compute the eigenvalues of the pair $(A + BF, E + BF)$ for F truncated to the displayed four digits. The resulting closed-loop eigenvalues are:

$$\{-1.999993, -0.50001, -0.999998, \infty, \infty\}$$

and thus are accurate to 5 decimal digits.

For the solution of DEAP3 we chose

$$\widetilde{A}_r = \text{diag}(-0.5, -1, -2)$$

and solved the minimization problem for J for the same values of α as above. The results are given in Table 3. The same tendencies for decreasing norms of F and increasing ill-conditioning with decreasing α values can be observed as in case of solving DEAP1. Because of increased parametric freedom, the proportional-derivative feedback achieves smaller norms for the same ranges of the condition numbers. This example also illustrate that small feedback gains alone is not sufficient for a robust numerical solution of the DEAP2.

α	$\|F\|_2$	$\kappa_2(X)$	$\kappa_2(Y)$
1	1.79	4.23	2.88
0.1	0.81	4.52	4.88
0.01	0.47	5.18	9.61
0.001	0.28	9.47	27.01
0.0001	0.17	19.86	96.39
0	0.0096	376.6	$5.48 \cdot 10^7$

Table 3: Results for DEAP3

For reference purpose we give the feedback matrix computed with $\alpha = 1$ for the pure robust DEAP3:

$$F = \begin{bmatrix} -0.0584 & 0.2600 & -0.3888 & 0.6921 & 0.0467 \\ 0.0647 & 0.0406 & 0.3480 & 0.0364 & 0.4027 \\ 0.3249 & 1.0523 & -0.3263 & 1.1542 & -0.0544 \end{bmatrix}$$

The robustness of the solution can be easily checked by computing the eigenvalues of the pair $(A + BF, E)$ with the elements of matrix F truncated to the displayed four digits. The resulting closed-loop eigenvalues are:

$$\{-1.99985, -0.999997, -0.500033, \infty, \infty\}$$

and thus are accurate to 4 decimal digits. This result is marginally better than that reported in [6] both with respect to the condition number of X as well as the magnitude of F. However practically the same robustness can be achieved for $\alpha = 0.01$ with a 5 times smaller magnitude of the feedback matrix F.

6 Conclusions

We focused on developing a reliable numerical approach to exploit the intrinsic non-uniqueness of the DEAP. One possibility to address the non-uniqueness is by formulating the DEAP as a minimum norm robust pole assignment problem. By using a convenient parametrization, a solution of the DEAP is sought by minimizing a special cost function expressing the weighted requirements for minimum Frobenius-norm of the feedback matrix and the minimum sensitivity of the closed-loop eigenvalues. The derived explicit expressions for the gradients of cost functions allow the use of standard gradient search based minimization techniques. The efficient evaluation of the cost functions and gradients is of paramount importance for the usefulness of the proposed approach. Using transformation techniques in conjunction with the solution of reduced generalized Sylvester equations is the main ingredient to achieve this goal. Further, it allows to address with practically no extra costs the partial pole assignment problem too. We believe that the proposed robust pole assignment approach is a viable way to solve large DEAPs in the perspective of the requirements formulated by recent sensitivity analysis results [9].

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