Aspects of Potential Vorticity Fluxes: Climatology and Impermeability

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ABSTRACT

Some aspects of the dynamics of generalized potential vorticity (PV) density $P = \boldsymbol{\omega} \cdot \nabla\chi$ are discussed with the main emphasis on P fluxes, where $\boldsymbol{\omega}_a$ is absolute vorticity and χ is a scalar. The impermeability theorem claims that there is no net P flux across a χ surface. Various forms of the flux are presented that mostly cross χ surfaces. As these fluxes are as dynamically relevant as the one chosen for the theorem, P fluxes through a surface element are inherently multivalued and there is no best choice on physical grounds. Nevertheless, the net P flux is unique for closed surfaces. This point is illustrated by P integrals over the volume between the earth's surface and an isentropic surface. Reanalysis data are used to present mean advective and some nonadvective P fluxes for $\chi = \theta$ in height coordinates. The extratropical tropopause appears to be supported by advective P fluxes. A satisfactorily closed P budget cannot, however, be presented.

1. Introduction

Potential vorticity (PV) in its various forms attracted considerable attention in the research community, partly because of its conservation in ideal flows and partly because of its invertibility, which allows to associate flow structures with separate anomalies of PV (e.g., Hoskins et al. 1985). Many aspects of the role of PV in the general atmospheric circulation have been investigated, where zonal and time mean fluxes were of primary interest (Yang et al. 1990; Hoskins 1991; Bartels et al. 1998; Schneider et al. 2003). Yang et al. (1990) and Bartels et al. (1998) derived eddy diffusion coefficients from meridional eddy PV fluxes. The meridional quasigeostrophic eddy PV flux is of central importance in the framework of the transformed Eulerian mean equations (e.g., Andrews et al. 1987). These fluxes have been presented repeatedly from data in the form of Eliassen–Palm flux divergences (e.g., Edmon et al. 1980). The divergence of the mean PV fluxes has to balance the heat and frictional forcing of PV. But while such budgets have been established previously for angular momentum and energy (Oort and Peixoto 1983), similar data evaluations have not been presented for PV despite its central role. For example, vertical eddy fluxes of PV are unknown, although there is little reason to believe that they are unimportant. It is a main purpose of this paper to provide a stepping-stone to a PV budget.

Ertel (1942) introduced the generalized potential vorticity:

$$Q = \frac{\boldsymbol{\omega}_a \cdot \nabla\chi}{\rho}, \qquad (1)$$

with absolute vorticity $\boldsymbol{\omega}_a$ and density ρ. As pointed out by Viúdez (2001), Q is a specific PV, but we will use the conventional term PV. Potential temperature θ is by far the most common and useful choice of the scalar χ. Moist forms of PV have also been explored (Schubert et al. 2001). The derivation of the prognostic equation for Q involves a χ equation of the form $d\chi/dt = \dot{\chi}$, where

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$\dot{\chi}$ is a known forcing function in space and time. The derivation proceeds by multiplying the gradient of the χ equation with $\boldsymbol{\omega}_a$ and the vorticity equation with $\nabla\chi$. The sum of both equations is the potential vorticity equation that, after invoking the equation of continuity and the nondivergence of vorticity, can be written as

$$\frac{\partial P}{\partial t} + \nabla\cdot(\mathbf{v}P) = \nabla\cdot(\boldsymbol{\omega}_a\dot{\chi}) + \nabla\cdot\left(\chi\nabla\frac{1}{\rho}\times\nabla p\right) + \nabla\cdot(\chi\nabla\times\mathbf{F}), \quad (2)$$

with PV density $P = \nabla\cdot(\boldsymbol{\omega}_a\chi)$ and $Q = P/\rho$, velocity $\mathbf{v} = (u, v, w)$, pressure p, and frictional acceleration $\mathbf{F}$. We may rewrite (2) as

$$\frac{\partial}{\partial t}P + \nabla\cdot\mathbf{j} = 0, \quad (3)$$

with P-flux vector $\mathbf{j}$ and components (j_1, j_2, j_3).

Although (2) is a general equation, the formulation of the divergences and vorticities depends on the coordinate system. We assume a standard (λ, φ, z) system with longitude λ, latitude φ, and height z. The local basis unit vectors are denoted by $\mathbf{e}_i$, with $\mathbf{e}_1$ pointing eastward, $\mathbf{e}_2$ northward, and $\mathbf{e}_3$ upward. Moreover, we employ the traditional approximation (e.g., Zdunkowski and Bott 2003), where Earth's rotational vorticity is $2\Omega\sin\varphi\mathbf{e}_3$, with $\Omega = 2\pi\,\mathrm{day}^{-1}$. Using the hydrostatic approximation, we obtain

$$P = (a\cos\varphi)^{-1}\omega_{a1}\frac{\partial\chi}{\partial\lambda} + a^{-1}\omega_{a2}\frac{\partial\chi}{\partial\varphi} + \omega_{a3}\frac{\partial\chi}{\partial z}, \quad (4)$$

with absolute vorticities

$$\omega_{a1} = -\frac{\partial v}{\partial z}, \quad \omega_{a2} = \frac{\partial u}{\partial z}, \quad \text{and}$$
$$\omega_{a3} = 2\Omega\sin\varphi + (a\cos\varphi)^{-1}\left[\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\varphi}(a\cos\varphi)\right]. \quad (5)$$

A budget can be established after applying a standard time and zonal averaging procedure to the fluxes $\mathbf{j}$ in (3):

$$\overline{b} = (2\pi T_r)^{-1}\int_0^{T_r}\int_0^{2\pi} b\,d\lambda\,dt, \quad (6)$$

with time interval T_r to obtain the mean fluxes

$$\overline{j}_2 = \overline{v}\,\overline{P} + \overline{v'P'} - \overline{\omega_{a2}\dot{\chi}} - \overline{\chi\frac{\partial F_1}{\partial z}} \quad \text{and}$$
$$\overline{j}_3 = \overline{w}\,\overline{P} + \overline{w'P'} - \overline{\omega_{a3}\dot{\chi}} - (a\cos\varphi)^{-1}\overline{\chi\left[\frac{\partial F_2}{\partial\lambda} - \frac{\partial}{\partial\varphi}(F_1\cos\varphi)\right]}, \quad (7)$$

where deviations are primed and F_1 and F_2 are the horizontal components of frictional acceleration. The first two rhs terms represent the advective P fluxes. Nonadvective fluxes are described by the remaining terms. The contribution of the pressure gradient term to (7) is deleted, because $\chi = \theta$ in our data evaluations.

Haynes and McIntyre (1987, hereafter HM87, 1990, hereafter HM90) published a theorem of obvious importance for PV fluxes, which has to be discussed before turning to the evaluation of PV budgets. Their impermeability theorem (IT) states that no net PV flux can cross an isentropic surface, while also being valid for generalized PV and χ surfaces. Thus, χ surfaces act as barriers for the fluxes, according to IT. It must have obvious consequences for a PV budget if the related fluxes are restricted this way. On the other hand, Truesdell and Toupin (1960) mention that there is no unique formulation of such fluxes (see also HM87 and HM90). This point was further explored by Schär (1993), Bretherton and Schär (1993), Bannon et al. (2003), and others, but these authors tend to agree that the flux chosen by HM87 is the flux to be accepted as the best choice on physical grounds (Bretherton and Schär 1993). However, Davies-Jones (2003) argues in favor of a different flux that also satisfies the IT. Recent textbooks (e.g., Vallis 2006; Mak 2011) present the IT as an important part of dynamic meteorology and discuss the reasons for its validity. On the other hand, the IT is not compatible with (2), as the flux $\theta\nabla\times\mathbf{F}$ in (2) for $\chi = \theta$ is normally not directed parallel to isentropic surfaces. For this to happen, the flux vector has to be orthogonal to $\nabla\theta$. Thus,

$$\nabla\theta\cdot(\nabla\times\mathbf{F}) = -(a\cos\varphi)^{-1}\frac{\partial\theta}{\partial\lambda}\frac{\partial F_2}{\partial z} + a^{-1}\frac{\partial\theta}{\partial\varphi}\frac{\partial F_1}{\partial z} + (a\cos\varphi)^{-1}\frac{\partial\theta}{\partial z}\left[\frac{\partial F_2}{\partial\lambda} - \frac{\partial}{\partial\varphi}(F_1\cos\varphi)\right] \quad (8)$$

would have to vanish, which is not true in general. Assuming meridionally sloping isentropes and frictional accelerations that depend only on z, we obtain $a^{-1}(\partial\theta/\partial\varphi)(\partial F_1/\partial z) \neq 0$ in (8).

Volume integrals of P are not exposed to these ambiguities. As stated by Gauss's theorem, their tendencies

$$\frac{d}{dt}\int_V P\,dV = \int_S (-\mathbf{j} + \mathbf{c}P)\cdot\mathbf{n}\,dS \qquad (9)$$

can be determined uniquely, even if there exist several versions of the flux $\mathbf{j}$ in (3), where V is the volume; S is its bounding surface moving with velocity $\mathbf{c}$; and $\mathbf{n}$ is the outward-pointing vector normal to S. Such integrals are of specific interest if parts of S are χ surfaces, as the IT postulates that the P fluxes cannot cross this surface.

This paper addresses the aforementioned issues and is organized as follows. We discuss the IT in section 2, followed by the presentation of a flux climatology in section 3. We conclude our presentation in section 4.

2. The impermeability theorem

The flux chosen by HM87 and HM90 for (3) is

$$\mathbf{j}_{\mathrm{HM}} = \mathbf{v}P - \boldsymbol{\omega}_a\dot{\chi} + \nabla\chi\times\mathbf{F}. \qquad (10)$$

The term $\nabla\cdot(\chi\nabla\times\mathbf{F})$ in (2) has been replaced by the equivalent, $-\nabla\cdot(\nabla\chi\times\mathbf{F})$, to arrive at (10). As above, the pressure gradient term has been deleted because of the choice of χ. The IT states that the combined contributions of $\mathbf{j}_{\mathrm{HM}}$ and the motion $\mathbf{c}$ to the surface integral in (9) vanish, even locally, for those parts of the bounding surface S where χ is constant. The authors were aware that P fluxes are ambiguous, but the related modifications were thought to be "needless complications" (McIntyre 2014). This view has been supported by Bretherton and Schär (1993), who argued that $\mathbf{j}_{\mathrm{HM}}$ is the best choice based on physical grounds. Their proof assumes that other choices of the nonadvective flux are of the form $\mathbf{a}\dot{\theta} + \mathbf{MF}$, where the vector $\mathbf{a}$ and the matrix $\mathbf{M}$ may depend in any way on the instantaneous position, velocity field, or thermodynamic variables (Bretherton and Schär 1993, p. 1836). In particular, $\mathbf{a} = -\boldsymbol{\omega}_a$ and $\mathbf{M} = \nabla\theta\times$ in (10). It is, however, obvious that the frictional flux in (2) cannot be brought into this form, because the operator $\nabla\theta\,\times$ contains derivatives. Furthermore, as we have seen above, the vector $\theta\nabla\times\mathbf{F}$ contains in general a component normal to isentropic surfaces.

Let us simplify this argument by turning to isentropic coordinates so that

$$(a\cos\varphi)^{-1}\left(-\frac{\partial\theta}{\partial\lambda}\bigg|_z\frac{\partial F_2}{\partial z} + \frac{\partial\theta}{\partial z}\frac{\partial F_2}{\partial\lambda}\bigg|_z\right) = (a\cos\varphi)^{-1}\left[-\frac{\partial\theta}{\partial\lambda}\bigg|_z\frac{\partial F_2}{\partial\theta}\frac{\partial\theta}{\partial z} + \frac{\partial\theta}{\partial z}\left(\frac{\partial F_2}{\partial\lambda}\bigg|_\theta + \frac{\partial F_2}{\partial\theta}\frac{\partial\theta}{\partial\lambda}\bigg|_z\right)\right] = (a\cos\varphi)^{-1}\frac{\partial F_2}{\partial\lambda}\bigg|_\theta\frac{\partial\theta}{\partial z}, \qquad (11)$$

and, similarly,

$$a^{-1}\frac{\partial\theta}{\partial\varphi}\frac{\partial F_1}{\partial z} - (a\cos\varphi)^{-1}\frac{\partial\theta}{\partial z}\frac{\partial(F_1\cos\varphi)}{\partial\varphi}\bigg|_z = -(a\cos\varphi)^{-1}\frac{\partial(F_1\cos\varphi)}{\partial\varphi}\bigg|_z\frac{\partial\theta}{\partial z}. \qquad (12)$$

Thus, the frictional flux normal to the isentropic surfaces is

$$\phi = \theta\nabla\times\mathbf{F}\cdot\frac{\nabla\theta}{|\nabla\theta|} = (a\cos\varphi)^{-1}\theta\left[\frac{\partial F_2}{\partial\lambda} - \frac{\partial(F_1\cos\varphi)}{\partial\varphi}\right]_\theta\frac{\partial\theta}{\partial z}|\nabla\theta|^{-1}. \qquad (13)$$

Hence, ϕ does not vanish in general. On the other hand, the frictional flux of $\mathbf{j}_{\mathrm{HM}}$ does not cross θ surfaces. Hence, a frictional flux through a surface element cannot be defined uniquely. One may prefer the form in (2) with respect to that in $\mathbf{j}_{\mathrm{HM}}$ because (2) is closer to the equation of motion, but that is a matter of opinion. In other words, the concept of a frictional flux through an area element is physically meaningless.

The pressure gradient term in (2) vanishes for $\chi = \theta$. In general, though, it can be written in three forms:

$$\nabla\cdot\left[\chi\left(\nabla\frac{1}{\rho}\times\nabla p\right)\right] = -\nabla\cdot\left(\frac{1}{\rho}\nabla\chi\times\nabla p\right) = \nabla\cdot\left(p\nabla\chi\times\nabla\frac{1}{\rho}\right), \qquad (14)$$

where the second and third term support the IT, but the first one does not. This would be relevant for $\chi = Q$, where the first term in (14) would be preferable and best reflects the physical meaning of the vorticity equation, as $\nabla\rho^{-1}\times\nabla p$ is part of this equation.

Even the term

$$\nabla\cdot(\boldsymbol{\omega}_a\dot{\chi}) = \nabla\cdot(\mathbf{v}_a\times\nabla\dot{\chi}), \qquad (15)$$

with absolute velocity $\mathbf{v}_a$ is problematic. For the lhs version, we can follow the argumentation of HM90, who found that the contributions of surface motion, advection, and forcing to the integrand in (9) vanish on χ surfaces. This proof does not work for the rhs version $\mathbf{v}_a \times \nabla\dot{\chi}$ of the flux, which in fact crosses χ surfaces. The lhs flux chosen by HM87 appears to be more attractive, as it simplifies the analysis and better reflects the χ equation.

This and similar statements above are thus a matter of opinion, as already stated for the frictional flux. There is no way to refute some of these fluxes as unphysical. Thus, the overall conclusion is that a total P flux through an area element cannot be evaluated. There is an exception, however, as the P flux through a closed surface is unique [see (9)].

Let us illustrate this point by looking at a volume V where the part S_θ of the surface S is a θ surface with $\theta = \theta_0$. Thus,

$$\int_V P\,dv = \theta_0 \int_{S_\theta} \boldsymbol{\omega}_a \cdot \mathbf{n}\,dS + \int_{S-S_\theta} \theta \boldsymbol{\omega}_a \cdot \mathbf{n}\,dS, \qquad (16)$$

and Stokes's theorem yields

$$\int_V P\,dv = \int_{S-S_\theta} (\theta - \theta_0)\boldsymbol{\omega}_a \cdot \mathbf{n}\,dS. \qquad (17)$$

Hence, we do not have to know the absolute vorticity on S_θ in order to calculate the contribution of S_θ to the surface integral (17).

Let us consider the volume enclosed by the earth's surface and the isentrope $\theta = \theta_0$, where θ_0 is chosen such that the θ_0 surface intersects the Northern Hemisphere, with V forming a dome centered at the North Pole [see Fig. 1 of Johnson (1989)]. After some manipulations, we obtain for a flat Earth

$$\frac{d}{dt}\int_V P\,dv = \int_{S-S_\theta} [-\omega_{aS}\dot{\theta}_S - (\nabla\theta_S \times \mathbf{F}_S)\cdot \mathbf{n}]\,dS, \qquad (18)$$

where the subscript S denotes surface values, and ω_{aS} is the vertical component of the absolute vorticity. The tendency can be evaluated unambiguously if we know the flow state on the surface area $S - S_\theta$. Surface heating implies negative tendencies, because ω_{aS} is positive. Friction has the same effect, at least for surface westerlies. The integral (18) describes the interaction of the air in V with the rest of the atmosphere and with the earth. If we follow the IT, there is no interaction with the outer atmosphere. If we accept (2), there is interaction both with the atmosphere outside of V and the earth. We simply cannot separate the flux through S into a part crossing S_θ and another one crossing $S - S_\theta$. The foregoing clarification of the concept of P fluxes will help us interpret flux observations.

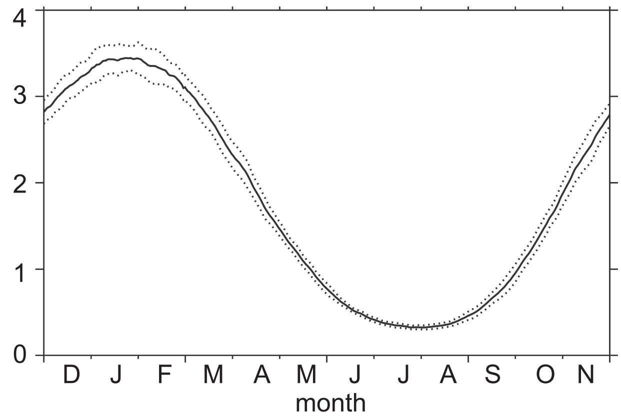

FIG. 1. Annual cycle of $\int_V P\,dv$ for the air volume underneath the isentropic surface $\theta = 285$ K in the Northern Hemisphere. Data are from ERA-Interim for 1979–2012. Shown are daily means, normalized by $8\pi\Omega a^2$. Dotted lines are standard deviations.

3. Mean *P* fluxes

We evaluate P fluxes on the basis of ERA-Interim data (Dee et al. 2011) for the years 1980–2013. This dataset contains winds, temperature, pressure, and $\dot{T}$ caused by physics. The calculations are performed for December–February (DJF) and June–August (JJA), respectively. The horizontal resolution used in this study is 2.25° × 2.25°, with time resolution of 6 h. We employed vertical interpolation to height θ surfaces at a vertical resolution of $\Delta z = 1000$ m ($\Delta\theta = 3$ K).

Let us first present an evaluation of the volume integral (17) for $\theta_0 = 285$ K, where we used ERA-Interim data for the years 1979–2012 (Dee et al. 2011). The results show a pronounced annual cycle with a maximum in winter, when the 285-K isentrope has its largest southward extension (Fig. 1). The integral (17) is positive, because $(\theta - \theta_0) < 0$, in general, and $\mathbf{n}$ points toward the center of Earth. This seasonal progression of the integral is mainly due to the heating (cooling) at the surface [see (18)], where the warming from January to July leads to a decrease of the area $S - S_\theta$, with a corresponding increase thereafter. The standard deviation is a few percent of the mean and must be due mainly to variations of the heating, but friction may play a role as well.

The results in Fig. 1 can be seen as a first step toward a climatology of isentropic volume integrals. We may evaluate P also for $\theta_0 = 290$ K and would find that the tendency for the layer between the two isentropic surfaces can be expressed as a surface integral like (18). There is, however, no tendency for layers that do not intersect the ground, because the first integral in (16) vanishes if S_θ is global.

Let us turn to the flux climatology. Most of our flux evaluations will be performed in height coordinates, where the averaged P budget is

$$(a\cos\varphi)^{-1}\frac{\partial}{\partial\varphi}(\bar{j}_2\cos\varphi)+\frac{\partial\bar{j}_3}{\partial z}=0. \qquad (19)$$

Isentropic coordinates are attractive, because P is equal to $(Q/\rho^*)\sim-(\zeta+f)$, with pseudodensity $\rho^*=-dp/d\theta$ so that one has only to look at the vorticity (Held and Schneider 1999). The replacement of ρ by ρ^* guarantees that a volume integral of ρ^* in isentropic coordinates yields mass.

The mean P equation in isentropic coordinates

$$-\overline{\upsilon(\zeta+f)}+\overline{\dot{\theta}\frac{\partial u}{\partial\theta}}-\overline{F_1}=0 \qquad (20)$$

equals the mean zonal velocity equation where differentiation as well as averaging are performed on θ surfaces. The frictional term cannot be evaluated because of the lack of data, so (20) reduces approximately to

$$-\overline{\upsilon'\zeta'}-f\overline{\upsilon}=0, \qquad (21)$$

where vertical advection is also neglected (Held and Schneider 1999; Schneider 2005). The simple form (21) is, however, only applicable above the surface zone that contains all isentropes that intersect the ground occasionally at a certain latitude. This zone has a depth of ~20 K in the tropics and ~80 K near the poles. An understanding of the flux budget in this zone is rather difficult, as the intersections move and generate new terms (Koh and Plumb 2004).

The validity of (21) is tested in Fig. 2. The Coriolis term $-f\overline{\upsilon}$ in Fig. 2a reflects the well-known hemispheric mean circulation in isentropic coordinates (Johnson 1989), with relatively intense equatorward flow near the ground and return flow aloft. Both terms are approximately symmetric with respect to the equator, with a switch of sign of the eddy transports in midlatitudes (Fig. 2b). The meridional flux of eddy vorticity is one order of magnitude smaller in the surface zone and even in the layers immediately above. Both terms are displayed in Fig. 2c, with equal contouring for $\theta\geq340$ K. It is evident that (21) is not useful in the belt 30°S–30°N and for $\theta\leq380$ K. Outside this tropical domain, the signs of both terms are mostly opposite, and the amplitudes match reasonably well. Thus, (21) is of reasonable quality in parts of the lower stratosphere and states that the meridional flux of absolute vorticity vanishes there, at least approximately.

We have to turn to height coordinates to learn more about P fluxes in the troposphere, where there are only orographic intersections of the coordinate surfaces. As stated above, the advective fluxes are not affected by ambiguities, whereas the nonadvective fluxes cannot be evaluated uniquely. We chose the fluxes in (7) without the frictional terms, which are not available. The fluxes $\overline{\upsilon}\,\overline{P}$ and $\overline{w}\,\overline{P}$ due to the mean flow reflect the mean circulation of the atmosphere. Because $\overline{P}\sim f\partial\overline{\theta}/\partial z$, one expects to recognize this circulation directly in the Northern Hemisphere (NH) and with reversed sign in the Southern Hemisphere (SH). The factor $\partial\overline{\theta}/\partial z$ is quite large in the stratosphere. We show the meridional mean flux in DJF in Fig. 3a, where all these features can be seen. The strength 10^{-7}–10^{-6} K s^{-2} of the flux corresponds to standard scaling estimates. The well-known cells of the Eulerian mean circulation can be seen quite clearly but are deeper because of the increase of the stability factor with height. The display in Figs. 3–6 is restricted to heights above 2 km because the evaluation of P requires one to compute centered vertical differences of θ and one-sided differences did not lead to satisfactory results.

The vertical mean fluxes (Fig. 3b) have the expected columnar structure with ascent (descent) near the equator in the NH (SH) and broad columnar descent (ascent) in the subtropics. The total mean flux $\overline{\upsilon P}\cos\varphi$ in DJF is displayed in Fig. 4a to aid the interpretation of the eddy fluxes. The total flux is rather symmetric in the lower troposphere. This symmetry disappears when we move upward. Antisymmetry is prevalent in the lower extratropical stratosphere. Thus, the seasonal and geographic differences of the hemispheres are dominating there. The extratropical tropopause region has convergent (divergent) fluxes in the NH (SH). The mean flow flux $\overline{\upsilon}\,\overline{P}$ is clearly dominating near the equator.

The eddy fluxes $\overline{\upsilon'P'}$ are displayed in Fig. 4 for DJF and JJA. Those for DJF are the difference of Figs. 4a and 3a. They are organized in vertical stripes with maximum fluxes on top. The flux is poleward in the domain 40°S–40°N and directed mainly toward the equator outside this belt. The flux in the NH is fairly weak in JJA, while that in the SH is less affected by the transition of the seasons. Unlike the eddy vorticity flux in Fig. 2b, $\overline{\upsilon'P'}$ is essentially antisymmetric with respect to the equator. This result would be difficult to accept on an aquaplanet without seasons, because both υ' and P' would be antisymmetric, and thus $\overline{\upsilon'P'}$ would be symmetric. Here, we lump all deviations together so that Figs. 4b and 4c contain contributions by stationary waves and long-term changes during the seasons. Further evaluations showed, however, that these contributions are not large.

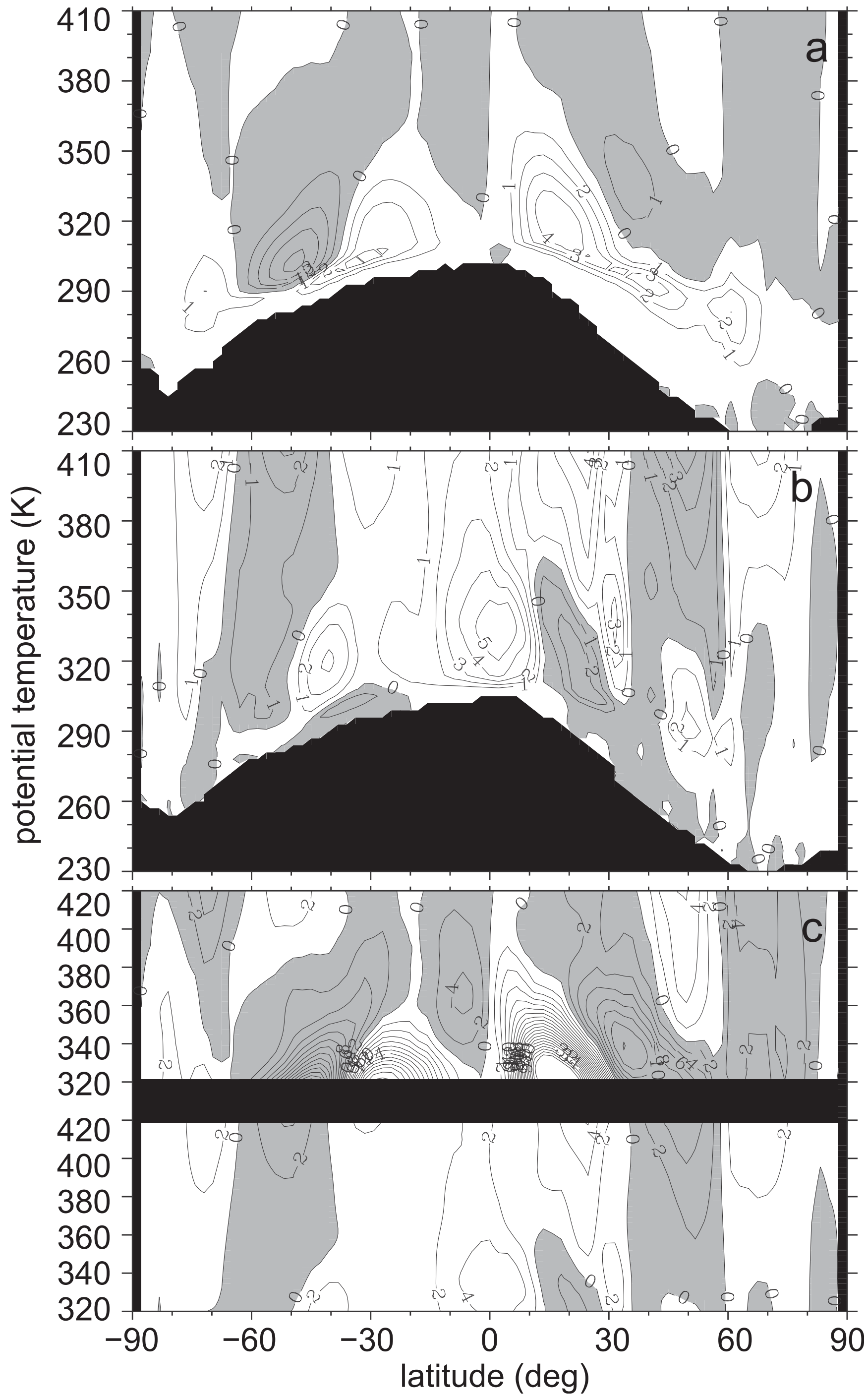

FIG. 2. Zonal integral and time mean of the terms (a) $-\overline{vf}$ (10^3 m^2 s^{-2}) and (b) $-\overline{v\zeta}$ (10^2 m^2 s^{-2}) in (21) isentropic coordinates for $230 < \theta < 410$ K in DJF. Also shown in (c) are both terms above 320 K with the same resolution as in (b). Negative values are in gray shading.

As stated above, it is customary to discuss meridional eddy PV fluxes in terms of quasigeostrophic theory, where $\overline{v'q'}$ is equal to the divergence of the Eliassen–Palm flux (e.g., Andrews et al. 1987) and q is the quasigeostrophic PV. The evaluation of this flux divergence shows a shallow layer of poleward flux near the ground in higher latitudes, with a deeper layer of return flow aloft that extends more and more southward with increasing height (e.g., Yang et al. 1990; Edmon et al. 1980). The fluxes are mainly symmetric with respect to the equator, in contrast to Fig. 4.

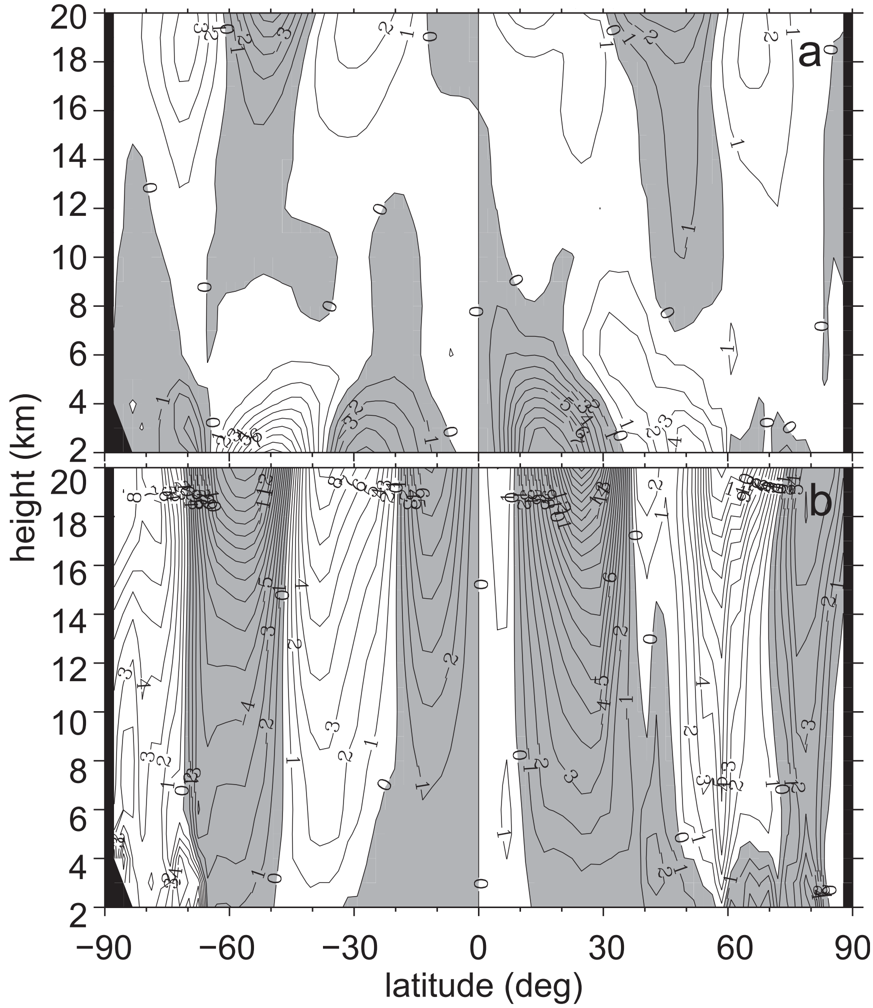

FIG. 3. Mean-flow P fluxes in DJF: (a) $\overline{v}\,\overline{P}\cos\varphi$ ($10^{-7}\,\mathrm{K\,s^{-2}}$) and (b) $\overline{w}\,\overline{P}$ ($10^{-9}\,\mathrm{K\,s^{-2}}$). Negative values are in gray shading.

The vertical eddy fluxes (Fig. 5) have a surprisingly simple pattern with upward fluxes almost everywhere in the NH troposphere. In addition, there are strong stratospheric downward fluxes in northern latitudes and a weaker deep equatorial upward branch near the equator. The NH fluxes are weaker in summer, and the separating line $\overline{w'P'} = 0$ is shifted upward. The SH pattern can essentially be obtained by a switch of sign, but the equatorial downward branch is quite narrow in JJA.

The position of the tropopause appears to be essential for an understanding of the flux patterns. The contours 2.5 and 3.5 PV units (PVU; 1 PVU = $10^{-6}\,\mathrm{K\,kg^{-1}\,m^2\,s^{-1}}$) are used to identify the tropopause in Figs. 4 and 5 (Hoinka 1998). Thus, convergence (divergence) of advective P fluxes near the tropopause in the NH (SH) implies a strengthening of the tropopause by synoptic systems. Such mechanisms have been discussed by Held (1982), Haynes et al. (2001), and others but have not been verified on the basis of data, although Hoskins (1991) found a switch of sign of the meridional eddy PV advection near the midlatitude tropopause. Some evidence for such processes is provided in Fig. 4a and can also be found in Fig. 5. For example, the polar NH tropopause in DJF is fairly level and located at a height of ~8–10 km. Vertical fluxes are directed upward below and downward above. The situation in the SH is not so clear cut.

The diabatic flux term is evaluated using the form in j_{HM} for the reasons given above. The horizontal component $-\overline{(\partial u/\partial z)\dot{\theta}}$ of the nonadvective fluxes is too small to be relevant. We obtain values 10^{-9}–$10^{-8}\,\mathrm{K\,s^{-2}}$ when compared to those of the advective fluxes $\geq 10^{-7}\,\mathrm{K\,s^{-2}}$. However, the vertical component $-f(\partial\overline{\theta}/\partial z)\dot{\theta}$ in Fig. 6 has the right order of magnitude. It represents the well-known global heating field with deep equatorial heating, low-level boundary heating, and narrow midlatitude towers of latent

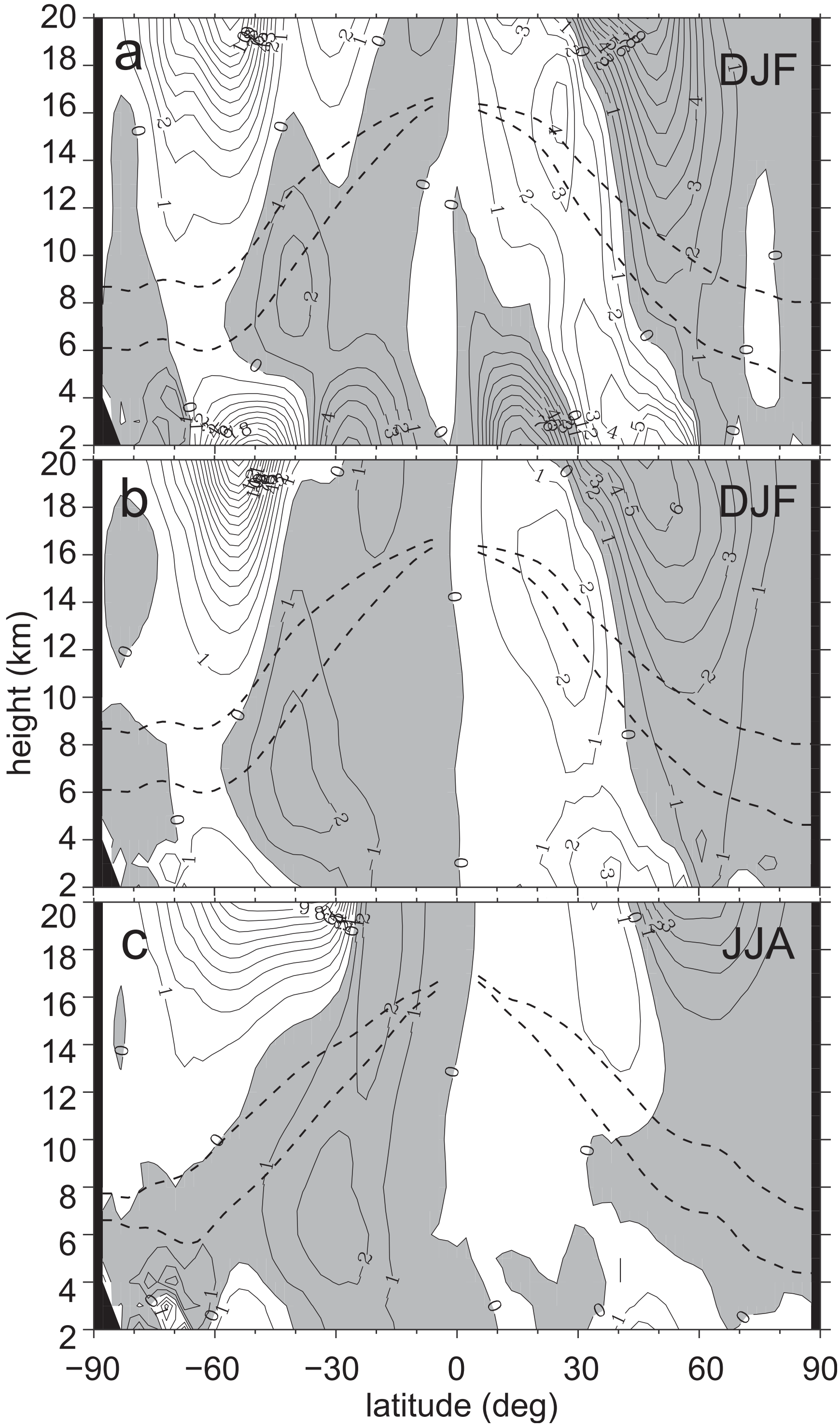

FIG. 4. Mean and eddy meridional P fluxes (10^{-7} K s^{-2}): (a) $\overline{vP}\cos\varphi$ in DJF and $\overline{v'P'}\cos\varphi$ in (b) DJF and (c) JJA. The broken lines indicate the tropopause region limited by 2.5 and 3.5 PVU. Negative values are in gray shading.

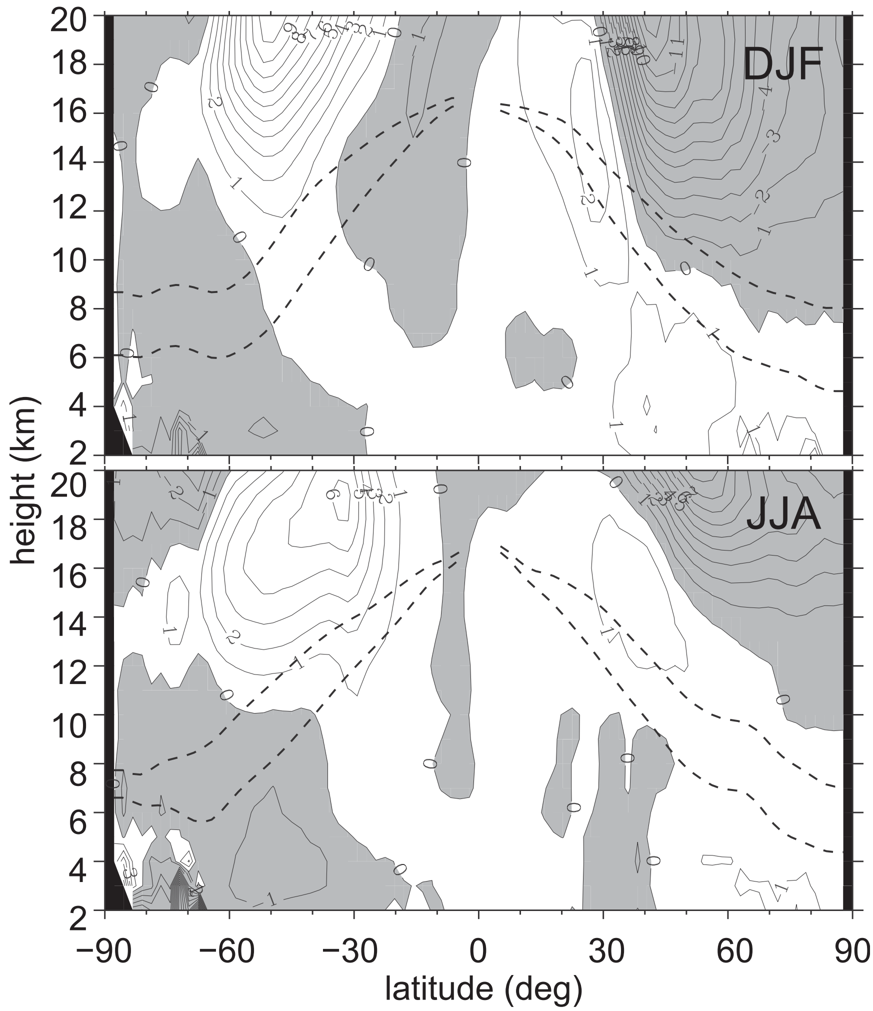

FIG. 5. Eddy P fluxes $\overline{w'P'}$ in (top) DJF and (bottom) JJA (10^{-9} K s^{-2}). The broken lines indicate the tropopause region limited by 2.5 and 3.5 PVU. Negative values are in gray shading.

heating. The flux is directed upward in the NH cooling regions.

To circumvent the ambiguity of the nonadvective fluxes, we evaluated the unique flux divergences. The resulting fields turned out to be quite noisy, particularly near the lower boundary (not shown). There is indeed eddy flux convergence (divergence) in the NH (SH) tropopause region, which appears to be partly balanced by the divergence (convergence) of the nonadvective fluxes, but the balance of all available divergences is not satisfactory. This may be partly because of the omission of frictional fluxes, but the divergences related to the mean flow flux are rather dominant. They have a columnar structure with the same (opposite) sign as $\overline{w}$ in the NH (SH). This suggests that this pattern is dominated by the component $\overline{w}(\partial\overline{P}/\partial z)$ of the mean flow divergence. It is not clear how this term is balanced by the divergences of the eddy fluxes or of the nonadvective flux.

4. Conclusions

Before drawing conclusions, we have to stress that this is not the first article with a critical look at the impermeability theorem. Danielsen's (1990) critique resulted in a clarification of several issues in HM90. Viúdez (1999) doubts the usefulness of the concept of a notional velocity $\mathbf{j}/P$ that "can be pictured as the velocity with which PVS molecules would move if the notional PVS were made of molecules" (HM90, p. 2036; PVS = PV substance = P). These molecules would have the choice between several velocities associated with the different flux vectors. Furthermore, there are obvious difficulties for situations in which $P = 0$ (Kieu and Zhang 2012).

It is a basic problem with P budgets that the P equation in (2) specifies the divergence of a flux but not the flux itself. This fact has been recognized before (Truesdell and Toupin 1960), but the full range of dynamically relevant

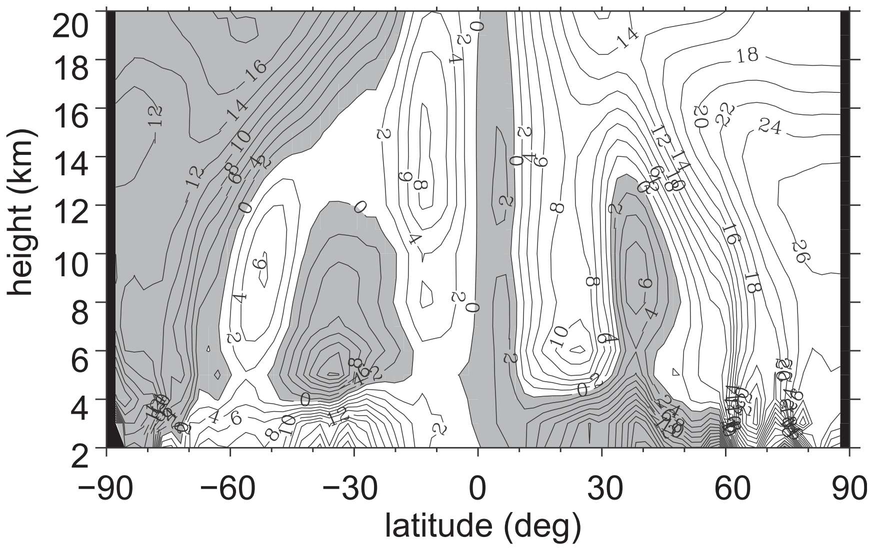

FIG. 6. Vertical nonadvective flux component $-\overline{(f+\zeta)\dot{\theta}}$ in DJF (10^{-10} K s^{-2}). Negative values are in gray shading.

options for the flux has not been explored. Although Bretherton and Schär (1993) and others were aware that fluxes can be defined that cross θ surfaces, it is the accepted view that noncrossing fluxes are to be preferred so that the IT is valid. However, after analyzing the available forms of the fluxes more closely, we find that one can also find a set of crossing fluxes that are dynamically equivalent to the noncrossing ones. The friction flux provides a prominent example in this context. Any preference for a specific form of the flux would have to be based on dynamic arguments. These are not available with respect to P fluxes. Hence, the P flux through an area element cannot be evaluated because of this inherent ambiguity.

The problem with this ambiguity is highlighted by the so-called electrodynamic analogy, which has been elaborated rather precisely by Schneider et al. (2003) in order to deal with boundary effects in PV dynamics. In this analogy, P corresponds to the electric charge density and the P flux $\mathbf{j}$ with the electric current density. However, while it would be absurd to have more than one electric current, we have to deal with a multitude of P fluxes in PV dynamics.

The tendency of volume integrals of PV density can be determined uniquely despite the multiplicity of the fluxes. We illustrated this point for atmospheric volumes underneath an isentrope. A choice of noncrossing fluxes would lead to the conclusion that the air in this volume exchanges PV only with the earth, while a choice of crossing fluxes would allow also for exchange with the atmosphere outside the volume.

Steps toward the evaluation of a climatological P budget have been undertaken, which were hampered by the lack of data on frictional processes. It turned out that the approximation (21) of a vanishing meridional absolute vorticity flux in isentropic coordinates is reasonably accurate in the lower stratosphere, but not in the troposphere, where intersections with the ground affect the analysis. Moreover, (21) neglects heating effects. These problems are overcome by turning to height coordinates and to a more complete presentation of P fluxes. The fluxes due to the mean circulation are fairly deep and not capped by the tropopause, as is the case with the mass circulation. The total meridional flux $\overline{vP}$ is symmetric with respect to the equator in the lower troposphere, where it mainly reflects the flux due to the mean circulation. Equatorward fluxes dominate in the extratropical lower stratosphere and are also found in the eddy flux pattern $\overline{v'P'}$. Their dynamic origin is not clear at the moment, because they are presumably not fully linked to the march of the seasons (see Figs. 4b,c). There is some support for the maintenance of the tropopause, both by the vertical eddy P fluxes and the mean meridional flux with convergence (divergence) in the NH (SH). The heating fluxes are mainly antisymmetric and reflect the well-known mean heating pattern. The heating fluxes are large enough to affect the total P budget substantially. It is, however, not possible to establish a reasonably closed P budget at the moment, because the available divergences are not accurate enough. In particular, the impact of the mean flow divergences on the budget is too large and noisy. Hence, further efforts are needed to establish such a budget.

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